

Graded Lie Superalgebras, Supertrace Formula, and Orbit Lie Superalgebras

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Abstract

Let Γ be a countable abelian semigroup and \mathcal{A} be a countable abelian group satisfying a certain finiteness condition. Suppose that a group G acts on a $(\Gamma \times \mathcal{A})$ -graded Lie superalgebra $\mathfrak{L} = \bigoplus_{(\alpha,a) \in \Gamma \times \mathcal{A}} \mathfrak{L}_{(\alpha,a)}$ by Lie superalgebra automorphisms preserving the $(\Gamma \times \mathcal{A})$ -gradation. In this paper, we show that the Euler-Poincaré principle yields the generalized denominator identity for \mathfrak{L} and derive a closed form formula for the supertraces $\text{str}(g|\mathfrak{L}_{(\alpha,a)})$ for all $g \in G$, $(\alpha, a) \in \Gamma \times \mathcal{A}$. We discuss the applications of our supertrace formula to various classes of infinite dimensional Lie superalgebras such as free Lie superalgebras and generalized Kac-Moody superalgebras. In particular, we determine the decomposition of free Lie superalgebras into a direct sum of irreducible $GL(n) \times GL(k)$ -modules, and the supertraces of the Monstrous Lie superalgebras with group actions. Finally, we prove that the generalized characters of Verma modules and the irreducible highest weight modules over a generalized Kac-Moody superalgebra \mathfrak{g} corresponding to the Dynkin diagram automorphism σ are the same as the usual characters of Verma modules and irreducible highest weight modules over the orbit Lie superalgebra $\check{\mathfrak{g}} = \mathfrak{g}(\sigma)$ determined by σ .

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1 Introduction

In the past three decades, the theory of infinite dimensional Lie algebras and their representations has been the focus of extensive research activities due to its rich and significant applications to many areas of mathematics and mathematical physics. The most well-known examples are *Kac-Moody algebras* ([27], [48]) and *generalized Kac-Moody algebras* ([5]), and a lot of exciting new discoveries have been made using the language of these infinite dimensional Lie algebras. For example, the *Macdonald identities* ([45]) were shown to be equivalent to the *denominator identities* of affine Kac-Moody algebras ([28]), and the *Moonshine Conjecture* was proved by investigating the structure of a special kind of generalized Kac-Moody algebra called the *Monster Lie algebra* ([6]).

On the other hand, since 1970's, the Lie superalgebras and their representations have emerged naturally as the fundamental algebraic structure behind several areas of mathematical physics. In [29], V. G. Kac gave a comprehensive presentation of the mathematical theory of Lie superalgebras, and obtained an important classification theorem for finite dimensional simple Lie superalgebras over algebraically closed fields of characteristic zero (see also [53]). The theories of Kac-Moody algebras and generalized Kac-Moody algebras have been extended to those of *Kac-Moody superalgebras* ([30]) and *generalized Kac-Moody superalgebras* ([47], [51]) following the outline given in [31]. In [34] and [35], V. G. Kac and M. Wakimoto developed the representation theory of affine Kac-Moody superalgebras, and demonstrated some interesting applications of affine Kac-Moody superalgebras to number theory.

In [37], S.-J. Kang derived a closed form *root multiplicity formula* for generalized Kac-Moody algebras, and by applying his formula to the Monster Lie algebra, he obtained some interesting recursive relations among the coefficients of the elliptic modular function j . The main idea of [37] is that the Euler-Poincaré principle for the Lie algebra homology can be interpreted as the denominator identity for Kac-Moody algebras or generalized Kac-Moody algebras, and once we are given the denominator identity, we can derive a root multiplicity formula. This idea has been extended to the general infinite dimensional graded Lie algebras to yield a closed form *dimension formula* for the homogeneous subspaces ([42]), and a lot of interesting applications of our formula to various classes of infinite dimensional Lie algebras have been investigated (see also [38] and [41]). In [32], V. G. Kac and S.-J. Kang exploited this idea further to derive a closed form *trace formula* for graded Lie algebras with group actions. In particular, by applying their trace formula to the Monster simple group acting on the Monster Lie algebra, they obtained a family of interesting recursive relations among the coefficients of the *Thompson series*.

The Thompson series are examples of *replicable functions* which are normalized q -series satisfying certain functional equations called the *replication formulae*. In [40], S.-J. Kang showed that, given the denominator identity for graded Lie superalgebras,

one can derive a closed form *superdimension formula* for the homogeneous subspaces, which can be considered as a unifying method of studying the structure of infinite dimensional Lie superalgebras. He discussed a lot of interesting applications of our superdimension formula to various classes of infinite dimensional Lie superalgebras such as free Lie superalgebras, generalized Kac-Moody superalgebras, and Monstrous Lie superalgebras. Furthermore, he characterized the replicable functions in terms of certain product identities, and determined the root multiplicities of the Monstrous Lie superalgebras associated with replicable functions.

In this paper, we combine the main ideas of [32] and [40] to derive a closed form *supertrace formula* for graded Lie superalgebras with group actions. More precisely, let Γ be a countable (usually infinite) abelian semigroup and \mathcal{A} be a countable (usually finite) abelian group such that every element $(\alpha, a) \in \Gamma \times \mathcal{A}$ can be written as a sum of elements in $\Gamma \times \mathcal{A}$ in only finitely many ways. Suppose that a group G acts on a graded Lie superalgebra $\mathfrak{L} = \bigoplus_{(\alpha, a) \in \Gamma \times \mathcal{A}} \mathfrak{L}_{(\alpha, a)}$ by Lie superalgebra automorphisms preserving the $(\Gamma \times \mathcal{A})$ -gradation. Then, in Section 2, we show that the Euler-Poincaré principle for the Lie superalgebra homology yields a product identity

$$\begin{aligned} \prod_{(\alpha, a) \in \Gamma \times \mathcal{A}} \exp \left(- \sum_{k=1}^{\infty} \frac{1}{k} \text{str}(g^k | \mathfrak{L}_{(\alpha, a)}) E^{k(\alpha, a)} \right) \\ = 1 - \sum_{(\alpha, a) \in \Gamma \times \mathcal{A}} \text{str}(g | H(\mathfrak{L})_{(\alpha, a)}) E^{(\alpha, a)}, \end{aligned}$$

which is called the *generalized denominator identity* for $g \in G$. By taking the logarithm of both sides of the generalized denominator identity and using Möbius inversion, we derive a closed form formula for the *supertraces* $\text{str}(g | \mathfrak{L}_{(\alpha, a)}) = \psi(a) \text{tr}(g | \mathfrak{L}_{(\alpha, a)})$ for all $g \in G$, $(\alpha, a) \in \Gamma \times \mathcal{A}$ (Theorem 2.4):

$$\text{str}(g | \mathfrak{L}_{(\alpha, a)}) = \sum_{\substack{d > 0 \\ (\alpha, a) = d(\tau, b)}} \frac{1}{d} \mu(d) W_{g^d}(\tau, b),$$

where $\psi(a)$ is the *sign map* defined on \mathcal{A} and $W_g(\tau, b)$ is the *Witt partition function* defined by (2.16).

In Section 3, we present the standard homology theory for Lie superalgebras. That is, we show that there exists a super-analogue of Koszul's complex for Lie superalgebras (Proposition 3.1). The k -th homology module $H_k(\mathfrak{L}, V)$ of the Lie superalgebra \mathfrak{L} with coefficients in V is defined by

$$H_k(\mathfrak{L}, V) = \text{Tor}_k^U(V, \mathbb{C}) = H_k(V \otimes_U M),$$

where $M = (M_k, \partial_k)$ is the super-analogue of Koszul's complex given by (3.1) and (3.2), and $U = U(\mathfrak{L})$ denotes the universal enveloping algebra of \mathfrak{L} . In particular, if $\mathfrak{L}(V)$ is the free Lie superalgebra generated by a superspace V , then we have $H_1(\mathfrak{L}(V)) = V$

and $H_k(\mathfrak{L}(V)) = 0$ for all $k \geq 2$. Even though our presentation is a straightforward generalization of the homology theory of Lie algebras (cf. [10], [15]), we decided to include the detailed treatment here, for we could not find it in any other literature.

Section 4 and Section 5 are devoted to the applications of our supertrace formula to free Lie superalgebras. In Section 4, we generalize the classical *Witt formula* for free Lie algebras to obtain a closed form supertrace formula for free Lie superalgebras with group actions. As an application, we compare the structure of the free Lie algebras and free Lie superalgebras generated by the same group representations (Proposition 4.2). In Section 5, we consider the free Lie superalgebra $\mathfrak{L}(V)$ generated by the natural $(k+l)$ -dimensional representation V of the general linear Lie superalgebra $gl(k, l)$, and determine the decomposition of $\mathfrak{L}(V)$ into a direct sum of irreducible $GL(k) \times GL(l)$ -modules (Proposition 5.3 and Proposition 5.4). The *Hook Schur functions* and the *Littlewood-Richardson coefficients* are the key ingredients of the decomposition.

In Section 6 and Section 7, we give a conjectural Kostant-type formula for the homology of the negative part of generalized Kac-Moody superalgebras, and derive a closed form supertrace formula for generalized Kac-Moody superalgebras with group actions preserving the root space decomposition (Theorem 6.5). Moreover, we apply our supertrace formula to determine the supertraces of the Monstrous Lie superalgebras with group actions preserving the root space decomposition (Proposition 7.1 and Corollary 7.2). We believe that the product identities characterizing the replicable functions ([40]) can be understood most naturally as the generalized denominator identities of some Mosntrous Lie superalgebra with a suitable group action.

In Section 8 and Section 9, we prove that the generalized characters of Verma modules and irreducible highest weight modules over a generalized Kac-Moody superalgebra \mathfrak{g} corresponding to the Dynkin diagram automorphism σ are the same as the usual characters of Verma modules and irreducible highest weight modules over the *orbit Lie superalgebra* $\check{\mathfrak{g}} = \mathfrak{g}(\sigma)$ determined by σ (Theorem 9.3 and Corollary 9.4). This fact was first proved in [13] for symmetrizable Kac-Moody algebras under the linking condition in solving the fixed point problem in conformal field theory and was generalized to generalized Kac-Moody algebras without the linking condition in [14]. In this paper, we also show that the generalized denominator identity for a generalized Kac-Moody superalgebra \mathfrak{g} for the Dynkin diagram automorphism σ is the same as the usual denominator identity for the orbit Lie superalgebra $\check{\mathfrak{g}} = \mathfrak{g}(\sigma)$ (Theorem 9.5). Our argument for generalized Kac-Moody superalgebras is similar to those in [13] and [14]. We observe that the generalized character of a Verma module can be written as the product part of a generalized denominator identity and derive an explicit formula for the traces of σ on each invariant subspaces (which are not the same as root spaces in general) of \mathfrak{g} in terms of the dimensions of some orbit Lie superalgebras. Therefore, we obtain a dimension formula for each homogeneous subspaces of the invariant subalgebra, which is also a generalized Kac-Moody superalgebra with a natural grading

arising from σ (Proposition 9.6).

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2 Graded Lie Superalgebras and Supertrace Formula

We recall some basic facts about Lie superalgebras ([29], [40], [53]). Let \mathcal{A} be a countable abelian group and suppose we have a bimultiplicative map $\theta : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{C}^\times$ satisfying

$$(2.1) \quad \begin{aligned} \theta(a+b, c) &= \theta(a, c)\theta(b, c), \\ \theta(a, b+c) &= \theta(a, b)\theta(a, c), \\ \theta(a, b)\theta(b, a) &= 1 \text{ for all } a, b \in \mathcal{A}. \end{aligned}$$

In particular, we have $\theta(a, a) = \pm 1$ for all $a \in \mathcal{A}$. The map θ satisfying the condition (2.1) is called a *coloring map on \mathcal{A}* . Define a map $\psi : \mathcal{A} \rightarrow \{\pm 1\}$ by $\psi(a) = \theta(a, a)$. Then ψ is a well-defined group homomorphism called the *sign map on \mathcal{A} with respect to θ* . Let $\mathcal{A}_0 = \{a \in \mathcal{A} \mid \psi(a) = 1\}$ and $\mathcal{A}_1 = \{a \in \mathcal{A} \mid \psi(a) = -1\}$. Then we have a decomposition $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$ and the elements of \mathcal{A} in \mathcal{A}_0 (resp. \mathcal{A}_1) are called *even* (resp. *odd*).

A θ -colored superspace is a pair (V, θ) , where $V = \bigoplus_{a \in \mathcal{A}} V_a$ is an \mathcal{A} -graded vector space and $\theta : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{C}^\times$ is a coloring map on \mathcal{A} . The elements of V_a are called *even* (resp. *odd*) if $\psi(a) = 1$ (resp. $\psi(a) = -1$). For each $a \in \mathcal{A}$, we define the *superdimension* of V_a to be

$$(2.2) \quad \text{sdim} V_a = \psi(a) \dim V_a.$$

Similarly, we define a θ -colored superalgebra to be a pair (U, θ) , where $U = \bigoplus_{a \in \mathcal{A}} U_a$ is an \mathcal{A} -graded associative algebra (i.e., $U_a U_b \subset U_{a+b}$ for all $a, b \in \mathcal{A}$) and $\theta : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{C}^\times$ is a coloring map on \mathcal{A} . The *direct sum* of \mathcal{A} -graded superalgebras is defined in the usual way, but, for θ -colored superalgebras $U = \bigoplus_{a \in \mathcal{A}} U_a$ and $U' = \bigoplus_{a' \in \mathcal{A}} U'_{a'}$, we define the *tensor product* of U and U' to be the θ -colored superspace $U \otimes U'$ with the natural \mathcal{A} -gradation

$$(U \otimes U')_{a+a'} = \text{Span}\{u \otimes u' \mid u \in U_a, u' \in U'_{a'}\}$$

and the multiplication given by

$$(u \otimes u')(v \otimes v') = \theta(a', b)(uv \otimes u'v')$$

for $u \in U_a, v \in U_b, u' \in U_{a'}, v' \in U_{b'}, a, a', b, b' \in \mathcal{A}$.

Definition 2.1 A θ -colored Lie superalgebra is a θ -colored superspace $\mathfrak{L} = \bigoplus_{a \in \mathcal{A}} \mathfrak{L}_a$ together with a bilinear operation $[\cdot, \cdot] : \mathfrak{L} \times \mathfrak{L} \rightarrow \mathfrak{L}$ satisfying

$$(2.3) \quad \begin{aligned} [\mathfrak{L}_a, \mathfrak{L}_b] &\subset \mathfrak{L}_{a+b}, \\ [x, y] &= -\theta(a, b)[y, x], \\ [x, [y, z]] &= [[x, y], z] + \theta(a, b)[y, [x, z]] \end{aligned}$$

for all $x \in \mathfrak{L}_a, y \in \mathfrak{L}_b, z \in \mathfrak{L}$ and $a, b \in \mathcal{A}$.

Let $\mathfrak{L}_0 = \bigoplus_{a \in \mathcal{A}_0} \mathfrak{L}_a$ and $\mathfrak{L}_1 = \bigoplus_{b \in \mathcal{A}_1} \mathfrak{L}_b$. Then we have a decomposition $\mathfrak{L} = \mathfrak{L}_0 \oplus \mathfrak{L}_1$, and the homogeneous elements of \mathfrak{L}_0 (resp. \mathfrak{L}_1) are called *even* (resp. *odd*).

The *universal enveloping algebra* of a θ -colored Lie superalgebra $\mathfrak{L} = \bigoplus_{a \in \mathcal{A}} \mathfrak{L}_a$ is the pair $(U(\mathfrak{L}), \iota)$, where $U(\mathfrak{L})$ is a θ -colored superalgebra and $\iota : \mathfrak{L} \rightarrow U(\mathfrak{L})$ is a linear mapping satisfying

$$\iota([x, y]) = \iota(x)\iota(y) - \theta(a, b)\iota(y)\iota(x) \quad \text{for } x \in \mathfrak{L}_a, y \in \mathfrak{L}_b$$

such that for any θ -colored superalgebra $U = \bigoplus_{a \in \mathcal{A}} U_a$ and a linear mapping $j : \mathfrak{L} \rightarrow U$ satisfying

$$j([x, y]) = j(x)j(y) - \theta(a, b)j(y)j(x) \quad \text{for } x \in \mathfrak{L}_a, y \in \mathfrak{L}_b,$$

there exists a unique homomorphism $\psi : U(\mathfrak{L}) \rightarrow U$ of θ -colored superalgebras satisfying $\psi \circ \iota = j$.

The uniqueness of the universal enveloping algebra $U(\mathfrak{L})$ can be proved in the usual way. For the existence, consider the tensor algebra $\mathcal{T}(\mathfrak{L}) = \bigoplus_{k=0}^{\infty} \mathfrak{L}^{\otimes k}$ of \mathfrak{L} and let \mathcal{R} be the ideal of $\mathcal{T}(\mathfrak{L})$ generated by the elements of the form

$$[x, y] - x \otimes y + \theta(a, b)y \otimes x \quad (x \in \mathfrak{L}_a, y \in \mathfrak{L}_b, a, b \in \mathcal{A}).$$

Then the quotient algebra $U(\mathfrak{L}) = \mathcal{T}(\mathfrak{L})/\mathcal{R}$ together with the natural mapping $\iota : \mathfrak{L} \rightarrow U(\mathfrak{L})$ is the universal enveloping algebra of \mathfrak{L} .

For the structure of the universal enveloping algebra $U(\mathfrak{L})$ of \mathfrak{L} , the following version of Poincaré-Birkhoff-Witt theorem is well-known.

Theorem 2.2 ([2], [29], [53])

Let \mathcal{A} be a countable abelian group with a coloring map $\theta : \mathcal{A} \times \mathcal{A} \rightarrow \mathbf{C}^\times$ and let $\mathfrak{L} = \bigoplus_{a \in \mathcal{A}} \mathfrak{L}_a$ be a θ -colored Lie superalgebra. Suppose $X = \{x_\alpha \mid \alpha \in \Lambda\}$ (resp. $Y = \{y_\beta \mid \beta \in \Omega\}$) is a homogeneous basis of the subspace $\mathfrak{L}_0 = \bigoplus_{a \in \mathcal{A}_0} \mathfrak{L}_a$ (resp. $\mathfrak{L}_1 = \bigoplus_{b \in \mathcal{A}_1} \mathfrak{L}_b$). Then the elements of the form

$$(2.4) \quad x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_k} y_{\beta_1} y_{\beta_2} \cdots y_{\beta_l} \quad \text{with } \alpha_1 \leq \cdots \leq \alpha_k, \beta_1 < \cdots < \beta_l$$

together with 1 form a basis of the universal enveloping algebra $U(\mathfrak{L})$ of \mathfrak{L} . \square

Remark. The theory of colored Lie superalgebras can be reduced to the theory of ordinary \mathbb{Z}_2 -graded Lie superalgebras by redefining the superbracket in a suitable way. More precisely, let $\mathfrak{L} = (\mathfrak{L}, [\ , \])$ be a θ -colored Lie superalgebra with a coloring map $\theta : \mathcal{A} \times \mathcal{A} \longrightarrow \mathbb{C}^\times$ and let $X = \{x_i \mid i = 1, 2, \dots\}$ be an ordered set of generators of \mathcal{A} . We define the bimultiplicative maps $\delta : \mathcal{A} \times \mathcal{A} \longrightarrow \mathbb{C}^\times$ and $\tau : \mathcal{A} \times \mathcal{A} \longrightarrow \mathbb{C}^\times$ by

$$\delta(a, b) = (-1)^{\psi(a)\psi(b)}\theta(b, a)$$

and

$$\tau(a, b) = \prod_{i < j} \delta(x_i, x_j)^{r_i s_j},$$

where $a = \sum r_i x_i$, $b = \sum s_i x_i \in \mathcal{A}$ ($r_i, s_i \in \mathbb{Z}$). We now introduce a new bracket $[\ , \]^\tau : \mathfrak{L} \times \mathfrak{L} \longrightarrow \mathfrak{L}$ defined by

$$[x, y]^\tau = \tau(a, b)[x, y]$$

for $x \in \mathfrak{L}_a$, $y \in \mathfrak{L}_b$. Then $\mathfrak{L}^\tau = (\mathfrak{L}, [\ , \]^\tau)$ becomes an ordinary \mathbb{Z}_2 -graded Lie superalgebra. Furthermore, there is a 1-1 correspondence between the set of \mathcal{A} -graded representations of the θ -colored Lie superalgebra $\mathfrak{L} = (\mathfrak{L}, [\ , \])$ and the set of \mathcal{A} -graded representations of the ordinary Lie superalgebra $\mathfrak{L}^\tau = (\mathfrak{L}, [\ , \]^\tau)$ (cf. [50], [54]).

Let \mathcal{A} be a countable (usually finite) abelian group with a coloring map θ , and let Γ be a countable (usually infinite) abelian semigroup such that every element $(\alpha, a) \in \Gamma \times \mathcal{A}$ can be written as a sum of elements in $\Gamma \times \mathcal{A}$ in only finitely many ways. Consider a $(\Gamma \times \mathcal{A})$ -graded θ -colored superspace $V = \bigoplus_{(\alpha, a) \in \Gamma \times \mathcal{A}} V_{(\alpha, a)}$ with $\dim V_{(\alpha, a)} < \infty$ for all $(\alpha, a) \in \Gamma \times \mathcal{A}$. Suppose that a group G acts on V preserving the $(\Gamma \times \mathcal{A})$ -gradation. We define the *generalized character* or the *graded trace* of V for $g \in G$ by

$$(2.5) \quad \text{ch}_g V = \sum_{(\alpha, a) \in \Gamma \times \mathcal{A}} \text{tr}(g|V_{(\alpha, a)}) e^{(\alpha, a)},$$

where $e^{(\alpha, a)}$ are the basis elements of the semi-group algebra $\mathbb{C}[\Gamma \times \mathcal{A}]$ with the multiplication given by $e^{(\alpha, a)} e^{(\beta, b)} = e^{(\alpha + \beta, a + b)}$.

On the other hand, we define the *supertrace* of the homogeneous subspace $V_{(\alpha, a)}$ for $g \in G$ by

$$(2.6) \quad \text{str}(g|V_{(\alpha, a)}) = \psi(a) \text{tr}(g|V_{(\alpha, a)}),$$

and introduce another basis elements of $\mathbb{C}[\Gamma \times \mathcal{A}]$ by setting $E^{(\alpha, a)} = \psi(a) e^{(\alpha, a)}$. Then it is easy to verify that $E^{(\alpha, a)} E^{(\beta, b)} = E^{(\alpha + \beta, a + b)}$. We now define the *generalized supercharacter* or the *graded supertrace* of V for $g \in G$ by

$$(2.7) \quad \text{sch}_g V = \sum_{(\alpha, a) \in \Gamma \times \mathcal{A}} \text{str}(g|V_{(\alpha, a)}) E^{(\alpha, a)}.$$

Note that $\text{sch}_g V$ is obtained from $\text{ch}_g V$ by replacing $\text{tr}(g|V_{(\alpha, a)}) = \psi(a) \text{str}(g|V_{(\alpha, a)})$ and $e^{(\alpha, a)} = \psi(a) E^{(\alpha, a)}$. Since $\psi(a)^2 = 1$, we have $\text{ch}_g V = \text{sch}_g V$.

Let $\mathfrak{L} = \bigoplus_{(\alpha,a) \in \Gamma \times \mathcal{A}} \mathfrak{L}_{(\alpha,a)}$ be a θ -colored $(\Gamma \times \mathcal{A})$ -graded Lie superalgebra with $\dim \mathfrak{L}_{(\alpha,a)} < \infty$ for all $(\alpha,a) \in \Gamma \times \mathcal{A}$ and suppose that a group G acts on \mathfrak{L} by Lie superalgebra automorphisms preserving the $\Gamma \times \mathcal{A}$ -gradation. In this work, using the Euler-Poincaré principle, which is equivalent to the *generalized denominator identity*, we will derive a closed form formula for the supertraces $\text{str}(g|\mathfrak{L}_{(\alpha,a)})$ for all $g \in G$, $(\alpha,a) \in \Gamma \times \mathcal{A}$.

Let \mathbb{C} be the trivial one dimensional \mathfrak{L} -module. The homology modules $H_k(\mathfrak{L}) = H_k(\mathfrak{L}, \mathbb{C})$ are determined from the following *standard complex*:

$$(2.8) \quad \cdots \rightarrow C_k(\mathfrak{L}) \xrightarrow{d_k} C_{k-1}(\mathfrak{L}) \xrightarrow{d_{k-1}} \cdots \rightarrow C_1(\mathfrak{L}) \xrightarrow{d_1} C_0(\mathfrak{L}) \rightarrow 0,$$

where $C_k(\mathfrak{L})$ are defined by

$$C_k(\mathfrak{L}) = \bigoplus_{p+q=k} \Lambda^p(\mathfrak{L}_0) \otimes S^q(\mathfrak{L}_1)$$

and the differentials $d_k : C_k(\mathfrak{L}) \rightarrow C_{k-1}(\mathfrak{L})$ are given by

$$\begin{aligned} d_k((x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots y_q)) &= \sum_{1 \leq s < t \leq p} (-1)^{s+t} ([x_s, x_t] \wedge x_1 \wedge \cdots \wedge \widehat{x}_s \wedge \cdots \wedge \widehat{x}_t \wedge \cdots \wedge x_p) \otimes (y_1 \cdots y_q) \\ &+ \sum_{s=1}^p \sum_{t=1}^q (-1)^s (x_1 \wedge \cdots \wedge \widehat{x}_s \wedge \cdots \wedge x_p) \otimes ([x_s, y_t] y_1 \cdots \widehat{y}_t \cdots y_q) \\ &- \sum_{1 \leq s < t \leq q} ([y_s, y_t] \wedge x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots \widehat{y}_s \cdots \widehat{y}_t \cdots y_q) \end{aligned}$$

for $k \geq 2$, $x_i \in \mathfrak{L}_0$, $y_j \in \mathfrak{L}_1$ and $d_1 = 0$ (cf. [10], [15]). In Section 3, we will give a more detailed treatment of the homology of \mathfrak{L} . Since the spaces $C_k(\mathfrak{L})$ and the homology modules $H_k(\mathfrak{L})$ inherit the $(\Gamma \times \mathcal{A})$ -gradation from that of \mathfrak{L} , the action of G and the generalized supercharacters of $C_k(\mathfrak{L})$ and $H_k(\mathfrak{L})$ are well-defined. Hence by the Euler-Poincaré principle, we obtain

$$(2.9) \quad \sum_{k=0}^{\infty} (-1)^k \text{sch}_g C_k(\mathfrak{L}) = \sum_{k=0}^{\infty} (-1)^k \text{sch}_g H_k(\mathfrak{L}) \quad \text{for all } g \in G.$$

Let

$$\begin{aligned} C(\mathfrak{L}) &= \sum_{k=0}^{\infty} (-1)^k C_k(\mathfrak{L}) = \mathbb{C} \oplus L \oplus C_2(\mathfrak{L}) \oplus \cdots, \\ \Lambda(\mathfrak{L}_0) &= \sum_{k=0}^{\infty} (-1)^k \Lambda^k(\mathfrak{L}_0) = \mathbb{C} \oplus \mathfrak{L}_0 \oplus \Lambda^2(\mathfrak{L}_0) \oplus \cdots, \\ S(\mathfrak{L}_1) &= \sum_{k=0}^{\infty} (-1)^k S^k(\mathfrak{L}_1) = \mathbb{C} \oplus \mathfrak{L}_1 \oplus S^2(\mathfrak{L}_1) \oplus \cdots, \\ H(\mathfrak{L}) &= \sum_{k=1}^{\infty} (-1)^{k+1} H_k(\mathfrak{L}) = H_1(\mathfrak{L}) \oplus H_2(\mathfrak{L}) \oplus H_3(\mathfrak{L}) \oplus \cdots, \end{aligned} \quad (2.10)$$

the alternating direct sum of superspaces. Then it is easy to see that $C(\mathfrak{L}) = \Lambda(\mathfrak{L}_0) \otimes S(\mathfrak{L}_1)$.

We recall the very basic theory of symmetric functions. Let x_1, \dots, x_k be indeterminates and for each $n \in \mathbb{N}$, define

$$(2.11) \quad \begin{aligned} p_n(x_1, \dots, x_k) &= \sum_{i=1}^k x_i^n, \\ h_n(x_1, \dots, x_k) &= \sum_{i_1 \leq \dots \leq i_n} x_{i_1} \cdots x_{i_n}, \\ e_n(x_1, \dots, x_k) &= \sum_{i_1 < \dots < i_n} x_{i_1} \cdots x_{i_n}, \end{aligned}$$

which are called the *n*th power sum, the *n*th completely symmetric function, and the *n*th elementary symmetric function, respectively. Note that the elementary symmetric function e_n is defined only for $n \leq k$. To compute the generalized supercharacters of $\Lambda(\mathfrak{L}_0)$ and $S(\mathfrak{L}_1)$, we recall the following fundamental lemma in the theory of symmetric functions:

Lemma 2.3 ([46])

$$(2.12) \quad \exp \left(\sum_{r \geq 1} \frac{p_r}{r} t^r \right) = \sum_{n \geq 0} h_n t^n = \frac{1}{\sum_{n \geq 0} (-1)^n e_n t^n}.$$

□

Let A be an $n \times n$ complex matrix with eigenvalues x_1, \dots, x_n . By Lemma 2.3, we have

$$(2.13) \quad \begin{aligned} \exp \left(\sum_{r \geq 1} \operatorname{tr}(A^r) \frac{t^r}{r} \right) &= \sum_{m \geq 0} \operatorname{tr}(A | S^m(\mathbb{C}^n)) t^m, \\ \exp \left(- \sum_{r \geq 1} \operatorname{tr}(A^r) \frac{t^r}{r} \right) &= \sum_{m \geq 0} (-1)^m \operatorname{tr}(A | \Lambda^m(\mathbb{C}^n)) t^m. \end{aligned}$$

Let $t = (t(\alpha, a))_{(\alpha, a) \in \Gamma \times \mathcal{A}}$ be a sequence of nonnegative integers indexed by $\Gamma \times \mathcal{A}$ with only finitely many nonzero terms, and set $|t| = \sum t(\alpha, a)$. Since the k th exterior power $\Lambda^k(\mathfrak{L}_0)$ is decomposed as

$$\Lambda^k(\mathfrak{L}_0) = \bigoplus_{|t|=k} \left(\bigotimes_{(\alpha, a) \in \Gamma \times \mathcal{A}_0} \Lambda^{t(\alpha, a)}(\mathfrak{L}_{(\alpha, a)}) \right),$$

we have

$$\operatorname{ch}_g \Lambda(\mathfrak{L}_0) = \sum_{k=0}^{\infty} (-1)^k \operatorname{ch}_g \Lambda^k(\mathfrak{L}_0)$$

$$\begin{aligned}
&= \prod_{(\alpha,a) \in \Gamma \times \mathcal{A}_0} \left(\sum_{m=0}^{\dim \mathfrak{L}_{(\alpha,a)}} (-1)^m \text{tr}(g | \Lambda^m(\mathfrak{L}_{(\alpha,a)})) e^{m(\alpha,a)} \right) \\
&= \prod_{(\alpha,a) \in \Gamma \times \mathcal{A}_0} \left(\sum_{m \geq 0} (-1)^m \text{tr}(g | \Lambda^m(\mathfrak{L}_{(\alpha,a)})) e^{m(\alpha,a)} \right) \\
&= \prod_{(\alpha,a) \in \Gamma \times \mathcal{A}_0} \exp \left(- \sum_{r=1}^{\infty} \frac{1}{r} \text{tr}(g^r | \mathfrak{L}_{(\alpha,a)}) e^{r(\alpha,a)} \right)
\end{aligned}$$

for all $g \in G$. Similarly, we have

$$\begin{aligned}
\text{ch}_g S(\mathfrak{L}_1) &= \sum_{k=0}^{\infty} (-1)^k \text{ch}_g S^k(\mathfrak{L}_1) \\
&= \prod_{(\alpha,a) \in \Gamma \times \mathcal{A}_1} \left(\sum_{m \geq 0} (-1)^m \text{tr}(g | S^m(\mathfrak{L}_{(\beta,b)})) e^{m(\beta,b)} \right) \\
&= \prod_{(\alpha,a) \in \Gamma \times \mathcal{A}_1} \left(\sum_{m \geq 0} \text{tr}(g | S^m(\mathfrak{L}_{(\beta,b)})) (-e^{(\beta,b)})^m \right) \\
&= \prod_{(\alpha,a) \in \Gamma \times \mathcal{A}_1} \exp \left(\sum_{r=1}^{\infty} \frac{(-1)^r}{r} \text{tr}(g^r | \mathfrak{L}_{(\beta,b)}) e^{r(\beta,b)} \right).
\end{aligned}$$

It follows that

$$\begin{aligned}
\text{ch}_g C(\mathfrak{L}) &= \sum_{k=0}^{\infty} (-1)^k \text{ch}_g C_k(\mathfrak{L}) = \text{ch}_g \Lambda(\mathfrak{L}_0) \cdot \text{ch}_g S(\mathfrak{L}_1) \\
&= \prod_{(\alpha,a) \in \Gamma \times \mathcal{A}_0} \exp \left(- \sum_{r=1}^{\infty} \frac{1}{r} \text{tr}(g^r | \mathfrak{L}_{(\alpha,a)}) e^{r(\alpha,a)} \right) \\
&\quad \times \prod_{(\beta,b) \in \Gamma \times \mathcal{A}_1} \exp \left(\sum_{r=1}^{\infty} \frac{(-1)^r}{r} \text{tr}(g^r | \mathfrak{L}_{(\beta,b)}) e^{r(\beta,b)} \right).
\end{aligned}$$

Replacing $E^{(\alpha,a)} = \psi(a) e^{(\alpha,a)}$ and $\text{str}(g | \mathfrak{L}_{(\alpha,a)}) = \psi(a) \text{tr}(g | \mathfrak{L}_{(\alpha,a)})$, the above identity yields

$$\text{sch}_g C(\mathfrak{L}) = \prod_{(\alpha,a) \in \Gamma \times \mathcal{A}} \exp \left(- \sum_{r=1}^{\infty} \frac{1}{r} \text{str}(g^r | \mathfrak{L}_{(\alpha,a)}) E^{r(\alpha,a)} \right).$$

Hence, by the Euler-Poincaré principle, we obtain the *generalized denominator identity* for the θ -colored graded Lie superalgebra $\mathfrak{L} = \bigoplus_{(\alpha,a) \in \Gamma \times \mathcal{A}} \mathfrak{L}_{(\alpha,a)}$:

$$(2.14) \quad \prod_{(\alpha,a) \in \Gamma \times \mathcal{A}} \exp \left(- \sum_{r=1}^{\infty} \frac{1}{r} \text{str}(g^r | \mathfrak{L}_{(\alpha,a)}) E^{r(\alpha,a)} \right) = 1 - \text{sch}_g H(\mathfrak{L}).$$

When $g = 1$, the identity element of G , we get the *denominator identity* for the Lie superalgebra \mathfrak{L} ([40]):

$$\prod_{\Gamma \times \mathcal{A}} (1 - E^{(\alpha, a)})^{\text{sdim} \mathfrak{L}_{(\alpha, a)}} = 1 - \text{sch} H(\mathfrak{L}).$$

Let $P(\Gamma \times \mathcal{A}) = \{(\alpha, a) \in \Gamma \times \mathcal{A} \mid \text{str}(g|H(\mathfrak{L})_{(\alpha, a)}) \neq 0\}$ and $\{(\alpha_i, a_j) \mid i, j = 1, 2, 3, \dots\}$ be an enumeration of $P(\Gamma \times \mathcal{A})$. For $(\tau, b) \in \Gamma \times \mathcal{A}$, set

$$(2.15) \quad T(\tau, b) = \{s = (s_{ij}) \mid s_{ij} \geq 0, \sum_{i,j} s_{ij}(\alpha_i, a_j) = (\tau, b)\},$$

the set of all partitions of (τ, b) into a sum of (α_i, a_j) 's, and define

$$(2.16) \quad W_g(\tau, b) = \sum_{s \in T(\tau, b)} \frac{(|s| - 1)!}{s!} \prod_{i,j} \text{str}(g|H(\mathfrak{L})_{(\alpha_i, a_j)})^{s_{ij}}.$$

We will call $W_g(\tau, b)$ the *Witt partition function* for $g \in G$. In the next theorem, using the generalized denominator identity (2.14), we derive a closed form formula for the supertraces $\text{str}(g|\mathfrak{L}_{(\alpha, a)})$ $((\alpha, a) \in \Gamma \times \mathcal{A})$ in terms of the Witt partition functions.

Theorem 2.4 *For $g \in G$ and $(\alpha, a) \in \Gamma \times \mathcal{A}$, we have*

$$(2.17) \quad \text{str}(g|\mathfrak{L}_{(\alpha, a)}) = \sum_{\substack{d > 0 \\ (\alpha, a) = d(\tau, b)}} \frac{1}{d} \mu(d) W_{g^d}(\tau, b).$$

Proof. By the generalized denominator identity (2.14), we have

$$\begin{aligned} & \exp \left(\sum_{(\alpha, a) \in \Gamma \times \mathcal{A}} \sum_{r=1}^{\infty} \frac{1}{r} \text{str}(g^r|\mathfrak{L}_{(\alpha, a)}) E^{r(\alpha, a)} \right) \\ &= \frac{1}{1 - \sum_{i,j=1}^{\infty} \text{str}(g|H(\mathfrak{L})_{(\alpha_i, a_j)}) E^{(\alpha_i, a_j)}}. \end{aligned}$$

Taking the logarithm and using the formal power series $\log(1 - x) = -\sum_{k=1}^{\infty} \frac{1}{k} x^k$, we obtain

$$\begin{aligned} & \sum_{(\alpha, a) \in \Gamma \times \mathcal{A}} \sum_{r=1}^{\infty} \frac{1}{r} \text{str}(g^r|\mathfrak{L}_{(\alpha, a)}) E^{r(\alpha, a)} \\ &= -\log \left(1 - \sum_{i,j=1}^{\infty} \text{str}(g|H(\mathfrak{L})_{(\alpha_i, a_j)}) E^{(\alpha_i, a_j)} \right)^{-1} \\ &= \sum_{m=1}^{\infty} \frac{1}{m} \left(\sum_{i,j=1}^{\infty} \text{str}(g|H(\mathfrak{L})_{(\alpha_i, a_j)}) E^{(\alpha_i, a_j)} \right)^m \end{aligned}$$

$$\begin{aligned}
&= \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\substack{s=(s_{ij}) \\ \sum s_{ij}=m}} \frac{(\sum s_{ij})!}{\prod s_{ij}!} \prod \text{str}(g|H(\mathfrak{L})_{(\alpha_i, a_j)})^{s_{ij}} E^{\sum s_{ij}(\alpha_i, a_j)} \\
&= \sum_{(\tau, b) \in \Gamma \times \mathcal{A}} \left(\sum_{s \in T(\tau, b)} \frac{(\sum s_{ij} - 1)!}{\prod s_{ij}!} \prod \text{str}(g|H(\mathfrak{L})_{(\alpha_i, a_j)})^{s_{ij}} \right) E^{(\tau, b)} \\
&= \sum_{(\tau, b) \in \Gamma \times \mathcal{A}} W_g(\tau, b) E^{(\tau, b)}.
\end{aligned}$$

It follows that

$$W_g(\tau, b) = \sum_{\substack{k>0 \\ (\tau, b)=k(\alpha, a)}} \frac{1}{k} \text{str}(g^k | \mathfrak{L}_{(\alpha, a)}).$$

Hence, by Möbius inversion, we obtain

$$\text{str}(g | \mathfrak{L}_{(\alpha, a)}) = \sum_{\substack{d>0 \\ d(\tau, b)=(\alpha, a)}} \frac{1}{d} \mu(d) W_{g^d}(\tau, b).$$

□

Remark. Note that our supertrace formula is a generalization of the *trace formula* for graded Lie algebras obtained in [32]. Also, when $g = 1$, our supertrace formula yields the *superdimension formula* for homogeneous subspaces of graded Lie superalgebras (cf. [40]).

3 Homology of Lie Superalgebras

The homology group of a Lie algebra is defined as the torsion group of its universal enveloping algebra and characterized by Koszul's complex. Similarly, we define the homology group of a graded Lie superalgebra as the torsion group of its universal enveloping algebra viewed as a supplemented algebra (cf. [10], [15]). In this section, we will show that there is a super-analogue of Koszul's complex for a graded Lie superalgebra and characterize its homology group. Our argument follows the framework given in [10] and [15].

Let $\mathfrak{L} = \mathfrak{L}_0 \oplus \mathfrak{L}_1$ be a Lie superalgebra and $U = U(\mathfrak{L})$ be its universal enveloping algebra. For each $k \geq 0$, define

$$(3.1) \quad C_k = C_k(\mathfrak{L}) = \bigoplus_{p+q=k} \Lambda^p(\mathfrak{L}_0) \otimes S^q(\mathfrak{L}_1).$$

Consider the following chain complex (M_k, ∂_k^\pm) ($k \geq -1$), where

$$M_k = \begin{cases} U \otimes_{\mathbb{C}} C_k(\mathfrak{L}) & \text{if } k \geq 0, \\ \mathbb{C} & \text{if } k = -1, \end{cases}$$

and the differentials $\partial_k : M_k \rightarrow M_{k-1}$ are given by

$$(3.2) \quad \begin{aligned} & \partial_k(u \otimes (x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots y_q)) \\ &= \sum_{1 \leq s < t \leq p} (-1)^{s+t} u \otimes ([x_s, x_t] \wedge x_1 \wedge \cdots \wedge \widehat{x}_s \wedge \cdots \wedge \widehat{x}_t \wedge \cdots \wedge x_p) \otimes (y_1 \cdots y_q) \\ &+ \sum_{s=1}^p \sum_{t=1}^q (-1)^s u \otimes (x_1 \wedge \cdots \wedge \widehat{x}_s \wedge \cdots \wedge x_p) \otimes ([x_s, y_t] y_1 \cdots \widehat{y}_t \cdots y_q) \\ &- \sum_{1 \leq s < t \leq q} u \otimes ([y_s, y_t] \wedge x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots \widehat{y}_s \cdots \widehat{y}_t \cdots y_q) \\ &+ \sum_{s=1}^p (-1)^{s+1} (u \cdot x_s) \otimes (x_1 \wedge \cdots \wedge \widehat{x}_s \wedge \cdots \wedge x_p) \otimes (y_1 \cdots y_q) \\ &+ (-1)^p \sum_{t=1}^q (u \cdot y_t) \otimes (x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots \widehat{y}_t \cdots y_q) \end{aligned}$$

for $k \geq 1$, ∂_0 is an augmentation map extracting the constant term, and $\partial_{-1} = 0$. Then it is easy to verify that $\partial_{k-1} \circ \partial_k = 0$.

Proposition 3.1 *The chain complex $M = (M_k, \partial_k)$ is a free resolution of the trivial 1-dimensional left U -module \mathbb{C} .*

Proof. **Case 1.** Suppose first that \mathfrak{L} is abelian. Then, by the Poincaré-Birkhoff-Witt Theorem, we have

$$(3.3) \quad U(\mathfrak{L}) \simeq S(\mathfrak{L}_0) \otimes \Lambda(\mathfrak{L}_1)$$

as a \mathbb{C} -vector space, and the differential maps are simplified to

$$\begin{aligned} & \partial_k(u \otimes (x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots y_q)) \\ &= \sum_{s=1}^p (-1)^{s+1} (u \cdot x_s) \otimes (x_1 \wedge \cdots \wedge \widehat{x}_s \wedge \cdots \wedge x_p) \otimes (y_1 \cdots y_q) \\ &+ (-1)^p \sum_{t=1}^q (u \cdot y_t) \otimes (x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots \widehat{y}_t \cdots y_q). \end{aligned}$$

We define the homotopy map $D : M_k \rightarrow M_{k+1}$ by

$$\begin{aligned} & D((u_1 \cdots u_s v_1 \cdots v_t) \otimes (x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots y_q)) \\ &= \sum_{i=1}^s (u_1 \cdots \widehat{u}_i \cdots u_s v_1 \cdots v_t) \otimes (u_i \wedge x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots y_q) \\ &= \sum_{j=1}^t (-1)^{p+t-j} (u_1 \cdots u_s v_1 \cdots \widehat{v}_j \cdots v_t) \otimes (x_1 \wedge \cdots \wedge x_p) \otimes (v_j y_1 \cdots y_q), \end{aligned}$$

where $u_i \in \mathfrak{L}_0$ and $v_j \in \mathfrak{L}_1$. Then it is easy to verify that

$$D\partial + \partial D = (p + q + s + t) 1_M,$$

which implies that our complex $M = (M_k, \partial_k)$ is exact in this case.

Case 2. Let \mathfrak{L} be any Lie superalgebra. Recall the construction of the universal enveloping algebra of \mathfrak{L} . Put $U^n(\mathfrak{L}) = \pi(\bigoplus_{k=0}^n \mathfrak{L}^{\otimes k})$, where $\pi : \mathcal{T}(\mathfrak{L}) \rightarrow U(\mathfrak{L})$ is the canonical projection map and $X_n^{(p)} = U^{p-n}(\mathfrak{L}) \otimes C_n(\mathfrak{L})$ for $0 \leq n \leq p$.

Consider the complex

$$(3.4) \quad 0 \longrightarrow X_p^{(p)} \xrightarrow{\partial_p} \cdots \xrightarrow{\partial_2} X_1^{(p)} \xrightarrow{\partial_1} X_0^{(p)} \xrightarrow{\epsilon} \mathbb{C} \longrightarrow 0.$$

We will use the induction on p to prove the exactness of (3.4), which would imply the exactness of our original sequence.

Note that the exactness of (3.4) is equivalent to that of the following sequence:

$$(3.5) \quad 0 \longrightarrow X_p^{(p)} \xrightarrow{\partial_p} \cdots \xrightarrow{\partial_2} X_1^{(p)} \xrightarrow{\partial_1} X_0^{(p)} / \mathbb{C} \longrightarrow 0.$$

For $p = 0, 1$, our assertion is clear. Suppose (3.5) is exact for $p - 1$. Consider the diagram

$$\begin{array}{ccccccc} & 0 & & & 0 & & 0 \\ & \downarrow & & & \downarrow & & \downarrow \\ 0 \rightarrow & 0 & \rightarrow & \cdots & \xrightarrow{\partial} & X_1^{(p-1)} & \xrightarrow{\partial} & X_0^{(p-1)} / \mathbb{C} & \rightarrow 0 \\ & \downarrow & & & \downarrow & & \downarrow \\ 0 \rightarrow & X_p^{(p)} & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & X_1^{(p)} & \xrightarrow{\partial} & X_0^{(p)} / \mathbb{C} & \rightarrow 0 \\ & \downarrow & & & \downarrow & & \downarrow \\ 0 \rightarrow & X_p^{(p)} & \xrightarrow{\bar{\partial}} & \cdots & \xrightarrow{\bar{\partial}} & X_1^{(p)} / X_1^{(p-1)} & \xrightarrow{\bar{\partial}} & X_0^{(p)} / X_0^{(p-1)} & \rightarrow 0. \\ & \downarrow & & & \downarrow & & \downarrow \\ & 0 & & & 0 & & 0 \end{array}$$

Note that

$$X_n^{(p)} / X_n^{(p-1)} \simeq (U^{p-n}(\mathfrak{L}) / U^{p-n-1}(\mathfrak{L})) \otimes C_n(\mathfrak{L}) \quad (0 \leq n \leq p-1)$$

as a \mathbb{C} -vector space, and that, by the Poincaré-Birkhoff-Witt Theorem,

$$U^{p-n}(\mathfrak{L}) / U^{p-n-1}(\mathfrak{L}) \simeq \bigoplus_{k+l=p-n} S^k(\mathfrak{L}_0) \otimes \Lambda^l(\mathfrak{L}_1)$$

by the canonical homomorphism and $X_p^{(p)} \simeq \mathbb{C} \otimes C_p(\mathfrak{L})$. Finally, the differentials in bottom row are given by

$$\begin{aligned} & \bar{\partial}_n(u \otimes (x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots y_q) + X_n^{(p-1)}) \\ (3.6) \quad &= \sum_{s=1}^p (-1)^{s+1} (u \cdot x_s) \otimes (x_1 \wedge \cdots \wedge \widehat{x_s} \wedge \cdots \wedge x_p) \otimes (y_1 \cdots y_q) \\ &+ (-1)^p \sum_{t=1}^q (u \cdot y_t) \otimes (x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots \widehat{y_t} \cdots y_q) + X_{n-1}^{(p-1)}. \end{aligned}$$

Thus the bottom row can be viewed as a subcomplex of the complex considered in Case 1, and hence it is exact. Therefore, the middle row is exact, which completes our induction argument.

Since $C(\mathfrak{L})$ is a \mathbb{C} -vector space, $U \otimes C(\mathfrak{L})$ is the direct sum of free U -modules. Hence, $U \otimes C(\mathfrak{L})$ is a free resolution of the 1-dimensional trivial left U -module \mathbb{C} . \square

Let V be a right U -module. The homology module $H_k(\mathfrak{L}, V)$ is defined to be

$$(3.7) \quad H_k(\mathfrak{L}, V) = \text{Tor}_k^U(V, \mathbb{C}) = H_k(V \otimes_U M),$$

where $M = (M_k, \partial_k)$ is the free resolution of the 1-dimensional trivial left U -module \mathbb{C} given by Proposition 3.1. Since we have

$$V \otimes_U M \cong V \otimes_U U \otimes_{\mathbb{C}} C_k(\mathfrak{L}) \cong V \otimes_{\mathbb{C}} C_k(\mathfrak{L}) \quad (k \geq 0),$$

and

$$V \otimes_U \mathbb{C} \cong V,$$

as \mathbb{C} -vector spaces, we obtain the following *standard complex* for the homology of Lie superalgebras:

$$(3.8) \quad \cdots \longrightarrow V \otimes_{\mathbb{C}} C_k(\mathfrak{L}) \xrightarrow{d_k} V \otimes_{\mathbb{C}} C_{k-1}(\mathfrak{L}) \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_1} V \otimes_{\mathbb{C}} C_0(\mathfrak{L}) \cong V \longrightarrow 0,$$

where the differentials $d_k = 1_V \otimes_U \partial_k : V \otimes_{\mathbb{C}} C_k(\mathfrak{L}) \longrightarrow V \otimes_{\mathbb{C}} C_{k-1}(\mathfrak{L})$ are given by

$$(3.9) \quad \begin{aligned} & d_k(v \otimes (x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots y_q)) \\ &= \sum_{1 \leq s < t \leq p} (-1)^{s+t} v \otimes ([x_s, x_t] \wedge x_1 \wedge \cdots \wedge \widehat{x}_s \wedge \cdots \wedge \widehat{x}_t \wedge \cdots \wedge x_p) \otimes (y_1 \cdots y_q) \\ &+ \sum_{s=1}^p \sum_{t=1}^q (-1)^s v \otimes (x_1 \wedge \cdots \wedge \widehat{x}_s \wedge \cdots \wedge x_p) \otimes ([x_s, y_t] y_1 \cdots \widehat{y}_t \cdots y_q) \\ &- \sum_{1 \leq s < t \leq q} v \otimes ([y_s, y_t] \wedge x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots \widehat{y}_s \cdots \widehat{y}_t \cdots y_q) \\ &+ \sum_{s=1}^p (-1)^{s+1} (v \cdot x_s) \otimes (x_1 \wedge \cdots \wedge \widehat{x}_s \wedge \cdots \wedge x_p) \otimes (y_1 \cdots y_q) \\ &+ (-1)^p \sum_{t=1}^q (v \cdot y_t) \otimes (x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots \widehat{y}_t \cdots y_q) \end{aligned}$$

for $v \in V$, $x_i \in \mathfrak{L}_0$, $y_j \in \mathfrak{L}_1$. In particular, when $V = \mathbb{C}$ is the trivial 1-dimensional right U -module, the standard complex (3.8) is reduced to the complex (2.8) given in Section 2.

As a corollary of Proposition 3.1, we obtain:

Corollary 3.2 *Let \mathcal{A} be a countable abelian group with a coloring map θ and $V = \bigoplus_{a \in \mathcal{A}} V_a$ be a θ -colored superspace. Let \mathfrak{L} be the free Lie superalgebra generated by V . Then we have*

$$(3.10) \quad H_k(\mathfrak{L}) = \begin{cases} V & \text{if } k = 1, \\ 0 & \text{if } k \geq 2. \end{cases}$$

Proof. Note that the universal enveloping algebra of \mathfrak{L} is isomorphic to the tensor algebra $\mathcal{T}(V)$ generated by V . Consider the following free resolution of the 1-dimensional trivial left $\mathcal{T}(V)$ -module \mathbb{C} :

$$0 \rightarrow \mathcal{T}(V) \otimes V \xrightarrow{\varphi} \mathcal{T}(V) \xrightarrow{\psi} \mathbb{C} \simeq \mathcal{T}(V)/\mathcal{T}(V) \otimes V \rightarrow 0,$$

where φ is the natural inclusion map (=multiplication map) and ψ is the augmentation map.

By tensoring with trivial right $\mathcal{T}(V)$ -module \mathbb{C} , we get

$$0 \rightarrow \mathbb{C} \otimes_{\mathcal{T}(V)} \mathcal{T}(V) \otimes V \xrightarrow{1 \otimes \varphi} \mathbb{C} \otimes_{\mathcal{T}(V)} \mathcal{T}(V) \xrightarrow{1 \otimes \psi} \mathbb{C} \otimes \mathbb{C} \rightarrow 0.$$

Since $1 \otimes \varphi$ is a zero map, $H_1(\mathfrak{L}) = \text{Ker}(1 \otimes \varphi) = \mathbb{C} \otimes_{\mathcal{T}(V)} \mathcal{T}(V) \otimes V \simeq V$, and $H_k(\mathfrak{L}) = 0$ for $k \geq 2$. \square

4 Free Lie Algebras and Free Lie Superalgebras

Let \mathcal{A} be a countable abelian group with a coloring map θ and Γ be a countable abelian semigroup satisfying the finiteness condition given in Section 2. Let $V = \bigoplus_{(\alpha, a) \in \Gamma \times \mathcal{A}} V_{(\alpha, a)}$ be a $(\Gamma \times \mathcal{A})$ -graded θ -colored superspace with finite dimensional homogeneous subspaces, and let $\mathfrak{L} = \bigoplus_{(\alpha, a) \in \Gamma \times \mathcal{A}} \mathfrak{L}_{(\alpha, a)}$ be the free Lie superalgebra generated by V . Suppose that a group G acts on V preserving the $(\Gamma \times \mathcal{A})$ -gradation. Then the action of G on V can be extended to \mathfrak{L} by Lie superalgebra automorphisms.

As we have seen in Section 3, we have $H_1(\mathfrak{L}) = V$ and $H_k(\mathfrak{L}) = 0$ for $k \geq 2$. Hence, for any $g \in G$, the generalized denominator identity for the free Lie superalgebra \mathfrak{L} is equal to

$$(4.1) \quad \prod_{(\alpha, a) \in \Gamma \times \mathcal{A}} \exp \left(- \sum_{k=1}^{\infty} \frac{1}{k} \text{str}(g^k | \mathfrak{L}_{(\alpha, a)}) E^{k(\alpha, a)} \right) \\ = 1 - \text{sch}_g V = 1 - \sum_{(\alpha, a) \in \Gamma \times \mathcal{A}} \text{str}(g | V_{(\alpha, a)}) E^{(\alpha, a)}.$$

For any $g \in G$, let $P(V, \Gamma \times \mathcal{A}) = \{(\alpha, a) \in \Gamma \times \mathcal{A} \mid \text{str}(g | V_{(\alpha, a)}) \neq 0\}$ and let $\{(\alpha_i, a_j) \mid i, j = 1, 2, 3, \dots\}$ be an enumeration of the set $P(V, \Gamma \times \mathcal{A})$. For each

$(\tau, b) \in \Gamma \times \mathcal{A}$, we denote by $T(\tau, b)$ the set of all partitions of (τ, b) into a sum of (α_i, a_j) 's as defined in (2.15), and let $W_g(\tau, b)$ be the Witt partition function as defined in (2.16). Then our supertrace formula (2.17) yields

$$(4.2) \quad \text{str}(g|\mathfrak{L}_{(\alpha,a)}) = \sum_{\substack{d>0 \\ (\alpha,a)=d(\tau,b)}} \frac{1}{d} \mu(d) \sum_{s \in T(\tau,b)} \frac{(|s|-1)!}{s!} \prod_{i,j} \text{str}(g^d|V_{(\alpha_i,a_j)})^{s_{ij}}.$$

Remark. The supertrace formula (4.2) is a generalization of the superdimension formula for the free Lie superalgebras given in [39] and [40].

Example 4.1 In this example, we will consider the simplest application of our supertrace formula to free Lie superalgebras. Consider the superspace $V = V_0 \oplus V_1$ with $\dim V_0 = r$ and $\dim V_1 = s$ for some $r, s \in \mathbb{Z}_{>0}$, and let \mathfrak{L} be the free Lie superalgebra generated by V . By setting $\deg v = 1$ for all $v \in V$, the free Lie superalgebra \mathfrak{L} has a $\mathbb{Z}_{>0}$ -gradation $\mathfrak{L} = \bigoplus_{n=1}^{\infty} \mathfrak{L}_n$ induced by V . Let $G = GL(V_0) \times GL(V_1) \cong GL(r) \times GL(s)$. For $(g, h) \in G$, we denote by $t_g = \text{tr}(g|V_0)$ and $t_h = \text{tr}(h|V_1)$. Since $P(V, \mathbb{Z}_{>0}) = \{1\}$ and $\text{str}((g, h)|V) = \text{tr}(g|V_0) - \text{tr}(h|V_1) = t_g - t_h$ for all $(g, h) \in G$, the Witt partition function $W_{(g,h)}(n)$ is given by

$$W_{(g,h)}(n) = \frac{(n-1)!}{n!} (t_g - t_h)^n = \frac{1}{n} (t_g - t_h)^n.$$

Hence, by the supertrace formula (4.2), we have

$$(4.3) \quad \text{str}((g, h)|\mathfrak{L}_n) = \frac{1}{n} \sum_{d|n} \mu(d) (t_g - t_h)^{\frac{n}{d}}. \quad \square$$

Next, we will discuss the relation of the free Lie algebra and the free Lie superalgebra generated by the same vector space $V = \bigoplus_{(\alpha,a) \in \Gamma \times \mathcal{A}} V_{(\alpha,a)}$ on which a group G acts preserving the $(\Gamma \times \mathcal{A})$ -gradation. By neglecting the coloring map θ on \mathcal{A} , consider V as a $(\Gamma \times \mathcal{A})$ -graded vector space, and let $L = \bigoplus_{(\alpha,a) \in \Gamma \times \mathcal{A}} L_{(\alpha,a)}$ be the free Lie algebra generated by V . Then the action of G on V can be extended to L by Lie algebra automorphisms. Hence, for any $g \in G$, the generalized ednominator identity for L is equal to

$$(4.4) \quad \prod_{(\alpha,a) \in \Gamma \times \mathcal{A}} \exp \left(- \sum_{k=1}^{\infty} \frac{1}{k} \text{tr}(g^k|L_{(\alpha,a)}) e^{k(\alpha,a)} \right) = 1 - \text{ch}_g V = 1 - \sum_{(\alpha,a) \in \Gamma \times \mathcal{A}} \text{tr}(g|V_{(\alpha,a)}) e^{(\alpha,a)}.$$

Since $E^{(\alpha,a)} = \psi(a)e^{(\alpha,a)}$ and $\text{str}(g|V_{(\alpha,a)}) = \psi(a)\text{tr}(g|V_{(\alpha,a)})$ for all $(\alpha, a) \in \Gamma \times \mathcal{A}$, the right-hand sides of (4.1) and (4.4) are the same. Moreover, note that, for an $n \times n$

complex matrix A with eigenvalues x_1, \dots, x_n , Lemma 2.1 implies

$$\begin{aligned} & \exp \left(\sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr}(A^k) t^k \right) \exp \left(\sum_{k=1}^{\infty} \frac{1}{k} (-1)^k \operatorname{tr}(A^k) t^k \right) \\ &= \prod \frac{1}{1 - x_i t} \prod \frac{1}{1 + x_i t} = \prod \frac{1}{1 - x_i^2 t^2} \\ &= \sum_{n=0}^{\infty} h_n(x_1^2, \dots, x_n^2) t^{2n} = \exp \left(\sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr}(A^{2k}) t^{2k} \right). \end{aligned}$$

Therefore, the left-hand side of (4.4) is equal to

$$\begin{aligned} & \prod_{(\alpha, a) \in \Gamma \times \mathcal{A}_0} \exp \left(- \sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr}(g^k | L_{(\alpha, a)}) e^{k(\alpha, a)} \right) \prod_{(\beta, b) \in \Gamma \times \mathcal{A}_1} \exp \left(- \sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr}(g^k | L_{(\beta, b)}) e^{k(\beta, b)} \right) \\ &= \prod_{(\alpha, a) \in \Gamma \times \mathcal{A}_0} \exp \left(- \sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr}(g^k | L_{(\alpha, a)}) E^{k(\alpha, a)} \right) \\ & \quad \times \prod_{(\beta, b) \in \Gamma \times \mathcal{A}_1} \exp \left(- \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \operatorname{tr}(g^k | L_{(\beta, b)}) E^{k(\beta, b)} \right) \\ &= \prod_{(\alpha, a) \in \Gamma \times \mathcal{A}_0} \exp \left(- \sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr}(g^k | L_{(\alpha, a)}) E^{k(\alpha, a)} \right) \\ & \quad \times \prod_{(\beta, b) \in \Gamma \times \mathcal{A}_1} \exp \left(\sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr}(g^k | L_{(\beta, b)}) E^{k(\beta, b)} \right) \\ & \quad \times \prod_{(\beta, b) \in \Gamma \times \mathcal{A}_1} \exp \left(- \sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr}(g^{2k} | L_{(\beta, b)}) E^{2k(\beta, b)} \right) \\ &= \prod_{(\alpha, a) \in \Gamma \times \mathcal{A}} \exp \left(- \sum_{k=1}^{\infty} \frac{1}{k} \psi(a) \operatorname{tr}(g^k | L_{(\alpha, a)}) E^{k(\alpha, a)} \right) \\ & \quad \times \prod_{\substack{(\alpha, a) = 2(\beta, b) \\ (\beta, b) \in \Gamma \times \mathcal{A}_1}} \exp \left(- \sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr}(g^{2k} | L_{(\beta, b)}) E^{k(\alpha, a)} \right). \end{aligned}$$

Hence, for any $g \in G$, the generalized denominator identity for the free Lie super-

algebra \mathfrak{L} is the same as

$$\begin{aligned}
(4.5) \quad & \prod_{(\alpha,a) \in \Gamma \times \mathcal{A}} \exp \left(- \sum_{k=1}^{\infty} \frac{1}{k} \text{str}(g^k | \mathfrak{L}_{(\alpha,a)}) E^{k(\alpha,a)} \right) \\
& = 1 - \sum_{(\alpha,a) \in \Gamma \times \mathcal{A}} \text{str}(g | V_{(\alpha,a)}) E^{(\alpha,a)} \\
& = \prod_{(\alpha,a) \in \Gamma \times \mathcal{A}} \exp \left(- \sum_{k=1}^{\infty} \frac{1}{k} \psi(a) \text{tr}(g^k | L_{(\alpha,a)}) E^{k(\alpha,a)} \right) \\
& \quad \times \prod_{\substack{(\alpha,a)=2(\beta,b) \\ (\beta,b) \in \Gamma \times \mathcal{A}_1}} \exp \left(- \sum_{k=1}^{\infty} \frac{1}{k} \text{tr}(g^{2k} | L_{(\beta,b)}) E^{k(\alpha,a)} \right),
\end{aligned}$$

which yields:

Proposition 4.2 *Let \mathcal{A} be a countable abelian group with a coloring map θ and Γ be a countable abelian semigroup satisfying the finiteness condition given in Section 2. For a $(\Gamma \times \mathcal{A})$ -graded superspace $V = \bigoplus_{(\alpha,a) \in \Gamma \times \mathcal{A}} V_{(\alpha,a)}$ with finite dimensional homogeneous subspaces, let $L = \bigoplus_{(\alpha,a) \in \Gamma \times \mathcal{A}} L_{(\alpha,a)}$ be the free Lie algebra generated by V , and let $\mathfrak{L} = \bigoplus_{(\alpha,a) \in \Gamma \times \mathcal{A}} \mathfrak{L}_{(\alpha,a)}$ be the free Lie superalgebra generated by V . Suppose that a group G acts on V preserving the $(\Gamma \times \mathcal{A})$ -gradation. Then, for any $g \in G$, we have*

$$(4.6) \quad \text{str}(g | \mathfrak{L}_{(\alpha,a)}) = \psi(a) \text{tr}(g | L_{(\alpha,a)}) + \sum_{\substack{(\alpha,a)=2(\beta,b) \\ (\beta,b) \in \Gamma \times \mathcal{A}_1}} \text{tr}(g^2 | L_{(\beta,b)}).$$

□

Remark. When $g = 1$, we recover the superdimension formula

$$(4.7) \quad \text{sdim } \mathfrak{L}_{(\alpha,a)} = \psi(a) \dim L_{(\alpha,a)} + \sum_{\substack{(\alpha,a)=2(\beta,b) \\ (\beta,b) \in \Gamma \times \mathcal{A}_1}} \dim L_{(\beta,b)},$$

which was obtained in [40].

5 Decomposition of Free Lie Superalgebras

In this section, we investigate the structure of the free Lie superalgebra generated by the natural representation of the *general linear Lie superalgebra* $gl(k, l)$. For $k, l \in \mathbb{Z}_{\geq 0}$, let $\mathfrak{L} = gl(k, l)$ be the space of all $(k + l) \times (k + l)$ matrices, and set

$$(5.1) \quad \mathfrak{L}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \middle| A \text{ is a } k \times k \text{ matrix and } D \text{ is an } l \times l \text{ matrix} \right\},$$

$$(5.2) \quad \mathfrak{L}_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \middle| B \text{ is a } k \times l \text{ matrix and } C \text{ is an } l \times k \text{ matrix} \right\}.$$

Then $\mathfrak{L} = \mathfrak{L}_0 \oplus \mathfrak{L}_1$ and \mathfrak{L} becomes a \mathbb{Z}_2 -graded Lie superalgebra called the *general linear Lie superalgebra* with the superbracket defined by

$$(5.3) \quad [X, Y] = XY - (-1)^{\alpha\beta} YX \quad \text{for } X \in \mathfrak{L}_\alpha, Y \in \mathfrak{L}_\beta, \alpha, \beta \in \mathbb{Z}_2.$$

A \mathbb{Z}_2 -graded superspace $V = V_0 \oplus V_1$ is an \mathfrak{L} -module if there is a \mathbb{Z}_2 -graded Lie superalgebra homomorphism

$$(5.4) \quad \phi : \mathfrak{L} \longrightarrow gl(V) \cong gl(k', l'), \quad \text{where } k' = \dim V_0, l' = \dim V_1.$$

For \mathfrak{L} -modules V^1, \dots, V^n , the tensor product $V^1 \otimes \dots \otimes V^n$ becomes an \mathfrak{L} -module with the action of \mathfrak{L} defined by

$$x \cdot (v_1 \otimes \dots \otimes v_n) = \sum_{i=1}^n (-1)^{\deg x (\sum_{j<i} \deg v_j)} v_1 \otimes \dots \otimes x \cdot v_i \otimes \dots \otimes v_n$$

for $x \in \mathfrak{L}$, $v_i \in V^i$.

We recall some of the basic theory of symmetric functions. A *partition* is any (finite or infinite) sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers in decreasing orders $\lambda_1 \geq \lambda_2 \geq \dots$ and containing only finitely many non-zero terms. The non-zero λ_i 's are called the *parts* of λ . The number of parts is the *length* of λ , denoted by $l(\lambda)$, and the sum of parts is the *weight* of λ , denoted by $|\lambda|$. If $|\lambda| = n$, then we say that λ is a *partition of n* , denote by $\lambda \vdash n$. We also use the notation which indicates the number of times each integer occurs as a part. That is, $\lambda = (1^{m_1}, 2^{m_2}, \dots)$ means that i appears as a part of λ exactly m_i times. Each partition λ uniquely determines a *Young diagram*. The *conjugate* of a partition λ is the partition λ' whose diagram is the transpose of that of λ (i.e., the diagram obtained by reflection in the main diagonal). There is a partial ordering on the set of partitions of n , called the *natural (partial) ordering* or the *dominance relation*, which is defined as follows:

$$\lambda \geq \mu \quad \text{if and only if} \quad \sum_{i=1}^k \lambda_i \geq \sum_{i=1}^k \mu_i \quad \text{for all } k \geq 1.$$

Let x_1, \dots, x_k be the indeterminates and λ a partition of n with $l(\lambda) \leq k$. A *Schur function* corresponding to λ is a symmetric function in x_1, \dots, x_k defined by

$$(5.5) \quad S_\lambda(x) = S_\lambda(x_1, \dots, x_k) = \frac{|x_i^{\lambda_j + n - j}|}{|x_i^{n-j}|}.$$

For a partition $\mu = (\mu_1, \mu_2, \dots)$, we define

$$(5.6) \quad p_\mu(x) = p_\mu(x_1, \dots, x_k) = \prod p_{\mu_i}(x_1, \dots, x_k),$$

where $p_{\mu_i}(x_1, \dots, x_k) = \sum x_j^{\mu_i}$ is the μ_i -th power sum. It is known that the set of Schur functions $\{S_\lambda(x) \mid l(\lambda) \leq k\}$ (resp. $\{p_\lambda(x) \mid l(\lambda) \leq k\}$) forms a \mathbb{Z} -basis (resp. \mathbb{Q} -basis) of the ring of symmetric functions in x_1, \dots, x_k , and $\{S_\lambda(x) \mid l(\lambda) \leq k, \lambda \vdash n\}$ (resp. $\{p_\lambda(x) \mid l(\lambda) \leq k, \lambda \vdash n\}$) forms a \mathbb{Z} -basis (resp. \mathbb{Q} -basis) of the subspace of symmetric functions in x_1, \dots, x_k with homogeneous degree n . The functions $S_\lambda(x)$ and $p_\lambda(x)$ are related as follows:

Lemma 5.1 ([46]) *For each partition ρ of n , we have*

$$(5.7) \quad p_\rho(x) = \sum_{\lambda \vdash n} \chi_\lambda^\rho S_\lambda(x),$$

where χ_λ^ρ is the character value of the irreducible representation of S_n corresponding to λ at the conjugacy class of type ρ . \square

A partition λ is called a (k, l) -hook partition if $\lambda_{k+1} \leq l$. Set $H(k, l; n) = \{\lambda \vdash n \mid \lambda_{k+1} \leq l\}$. For convenience, we will often write H for $H(k, l; n)$. For $\lambda \in H(k, l; n)$, a (k, l) -hook Schur function in $x_1, \dots, x_k, y_1, \dots, y_l$ corresponding to λ is a function defined by

$$(5.8) \quad HS_\lambda(x, y) = HS_\lambda(x_1, \dots, x_k, y_1, \dots, y_l) = \sum_{\mu < \lambda} S_\mu(x) S_{\lambda'/\mu'}(y),$$

where $S_{\lambda'/\mu'}(y)$ is the skew Schur function corresponding to λ', μ' (cf. [8], [46], [52]). Equivalently,

$$(5.9) \quad HS_\lambda(x, y) = \sum_{\mu < \lambda} S_\mu(x) \left(\sum_{\nu} N_{\mu'\nu}^{\lambda'} S_\nu(y) \right),$$

where $N_{\mu'\nu}^{\lambda'}$ is the *Littlewood-Richardson coefficient* corresponding to λ', μ', ν . The combinatorial interpretation of the Littlewood-Richardson coefficients is given as follows: for given partitions λ, μ, ν satisfying $|\lambda| = |\mu| + |\nu|$ and $\lambda_i \geq \mu_i$ for all i , $N_{\mu\nu}^\lambda$ is the number of ways the Young diagram for μ can be expanded to the Young diagram for λ by a *strict ν -expansion* (if the above conditions do not hold, $N_{\mu\nu}^\lambda = 0$). For a partition $\nu = (\nu_1, \dots, \nu_k)$, a ν -expansion of a Young diagram for μ is obtained by first adding ν_1 boxes to the Young diagram for μ with no two boxes in the same column and putting the integer 1 in each of these ν_1 boxes; then adding similarly ν_2 boxes with a 2, continuing until finally ν_k boxes are added with the integer k . The expansion is called *strict* if, when the integers in the boxes are listed from right to left, starting with the top row and working down, and one looks at the first t entries in this list (for any t between 1 and $|\nu|$), each integer p between 1 and $k-1$ occurs at least as many times as the next integer $p+1$.

In [8], Berele and Regev showed that, for a \mathbb{Z}_2 -graded superspace $V = V_0 \oplus V_1$ viewed as a $gl(k, l)$ -module ($k = \dim V_0, l = \dim V_1$), $V^{\otimes n}$ is completely reducible

as a $gl(k, l)$ -module and its irreducible components are parametrized by (k, l) -hook partitions λ , denoted by V_λ .

Let $\mathfrak{L}(V)$ be the free Lie superalgebra generated by V . Set $\mathfrak{L}(V)_{(n,0)} = \mathfrak{L}(V) \cap (V^{\otimes n})_0$, $\mathfrak{L}(V)_{(n,1)} = \mathfrak{L}(V) \cap (V^{\otimes n})_1$ and $\mathfrak{L}_n(V) = \mathfrak{L}(V)_{(n,0)} \oplus \mathfrak{L}(V)_{(n,1)}$. Thus $\mathfrak{L}(V)$ has an $(\mathbb{N} \times \mathbb{Z}_2)$ -gradation: $\mathfrak{L}(V) = \bigoplus_{\mathbb{N} \times \mathbb{Z}_2} \mathfrak{L}(V)_{(n,\alpha)}$. The homogeneous subspace $\mathfrak{L}_n(V)$ is a $gl(k, l)$ -submodule of $V^{\otimes n}$, where the action of $x \in gl(k, l)$ induced from $V^{\otimes n}$ is given by

$$\begin{aligned} & x \cdot [v_1, [v_2, [\cdots [v_{n-1}, v_n] \cdots]]] \\ &= \sum_{i=1}^n (-1)^{\deg x (\sum_{j < i} \deg v_j)} [v_1, [\cdots [xv_i [\cdots [v_{n-1}, v_n] \cdots]]]], \end{aligned}$$

and it is completely reducible with its irreducible components parametrized by $\lambda's \in H$.

From now on, we simply write \mathfrak{L}_n , $\mathfrak{L}_{(n,0)}$ and $\mathfrak{L}_{(n,1)}$ for $\mathfrak{L}_n(V)$, $\mathfrak{L}(V)_{(n,0)}$ and $\mathfrak{L}(V)_{(n,1)}$, respectively. For each $\lambda \in H(k, l; n)$, let c_λ be the multiplicity of V_λ in \mathfrak{L}_n . To compute c_λ , we would like to apply our supertrace formula. By exponentiating the action of the Lie algebra $gl(k, l)_0 = gl(k) \times gl(l)$ on V , the group $GL(k) \times GL(l)$ acts on V , and hence $V^{\otimes n}$ and $\mathfrak{L}_n(V)$ become $GL(k) \times GL(l)$ -modules. We first recall:

Proposition 5.2 ([8]) *Let $\lambda \in H(k, l; n)$ and V_λ be the corresponding irreducible representation of $gl(k, l)$. Suppose that $g = (g_0, g_1) \in GL(k) \times GL(l)$ has eigenvalues x_1, \dots, x_k for g_0 and y_1, \dots, y_l for g_1 . Then we have*

$$(5.10) \quad \text{tr}(g|V_\lambda) = HS_\lambda(x, y).$$

□

Our supertrace formula (4.2) for free Lie superalgebras yields

$$(5.11) \quad \text{str}(g|\mathfrak{L}_{(n,\alpha)}) = \sum_{d|(n,\alpha)} \frac{1}{d} \mu(d) \sum_{s \in T((n,\alpha)/d)} \frac{(|s| - 1)!}{s!} \text{str}(g_0^d|V_0)^{s_0} \text{str}(g_1^d|V_1)^{s_1},$$

where

$$\begin{aligned} T(m, 0) &= \{ (s_0, s_1) \mid s_0 + s_1 = m, \ s_1 \equiv 0 \pmod{2} \}, \\ T(m, 1) &= \{ (s_0, s_1) \mid s_0 + s_1 = m, \ s_1 \equiv 1 \pmod{2} \}. \end{aligned}$$

Hence, for $g = (g_0, g_1) \in GL(k) \times GL(l)$, we have

$$\begin{aligned}
\text{tr}(g|\mathfrak{L}_{(n,0)}) &= \text{str}(g|\mathfrak{L}_{(n,0)}) \\
&= \sum_{d|(n,0)} \frac{1}{d} \mu(d) \sum_{s \in T((n,0)/d)} \frac{(|s| - 1)!}{s!} \text{str}(g_0^d|V_0)^{s_0} \text{str}(g_1^d|V_1)^{s_1} \\
&= \sum_{d|(n,0)} \frac{1}{d} \mu(d) \sum_{s \in T((n,0)/d)} \frac{(|s| - 1)!}{s!} (-1)^{s_1} p_d(x)^{s_0} p_d(y)^{s_1} \\
&= \sum_{d|(n,0)} \frac{1}{d} \mu(d) \sum_{s \in T((n,0)/d)} \frac{(|s| - 1)!}{s!} (-1)^{s_1} \left(\sum_{\lambda \vdash ds_0} \chi_\lambda^{(d^{s_0})} S_\lambda(x) \right) \left(\sum_{\mu \vdash ds_1} \chi_\mu^{(d^{s_1})} S_\mu(y) \right) \\
&= \sum_{d|(n,0)} \frac{1}{d} \mu(d) \sum_{s \in T((n,0)/d)} \frac{(|s| - 1)!}{s!} (-1)^{s_1} \left(\sum_{\substack{\lambda \vdash ds_0 \\ \mu \vdash ds_1}} \chi_\lambda^{(d^{s_0})} \chi_\mu^{(d^{s_1})} S_\lambda(x) S_\mu(y) \right),
\end{aligned}$$

and

$$\begin{aligned}
\text{tr}(g|\mathfrak{L}_{(n,1)}) &= -\text{str}(g|\mathfrak{L}_{(n,1)}) \\
&= - \sum_{\substack{d|n \\ d:\text{odd}}} \frac{1}{d} \mu(d) \sum_{s \in T((n,1)/d)} \frac{(|s| - 1)!}{s!} \text{str}(g_0^d|V_0)^{s_0} \text{str}(g_1^d|V_1)^{s_1} \\
&= - \sum_{\substack{d|n \\ d:\text{odd}}} \frac{1}{d} \mu(d) \sum_{s \in T((n,1)/d)} \frac{(|s| - 1)!}{s!} (-1)^{s_1} \text{tr}(g_0^d|V_0)^{s_0} \text{tr}(g_1^d|V_1)^{s_1} \\
&= \sum_{\substack{d|n \\ d:\text{odd}}} \frac{1}{d} \mu(d) \sum_{s \in T((n,1)/d)} \frac{(|s| - 1)!}{s!} p_d(x)^{s_0} p_d(y)^{s_1} \\
&= \sum_{\substack{d|n \\ d:\text{odd}}} \frac{1}{d} \mu(d) \sum_{s \in T((n,1)/d)} \frac{(|s| - 1)!}{s!} \left(\sum_{\lambda \vdash ds_0} \chi_\lambda^{(d^{s_0})} S_\lambda(x) \right) \left(\sum_{\mu \vdash ds_1} \chi_\mu^{(d^{s_1})} S_\mu(y) \right) \\
&= \sum_{\substack{d|n \\ d:\text{odd}}} \frac{1}{d} \mu(d) \sum_{s \in T((n,1)/d)} \frac{(|s| - 1)!}{s!} \left(\sum_{\substack{\lambda \vdash ds_0 \\ \mu \vdash ds_1}} \chi_\lambda^{(d^{s_0})} \chi_\mu^{(d^{s_1})} S_\lambda(x) S_\mu(y) \right).
\end{aligned}$$

It follows that

$$\begin{aligned}
(5.12) \quad & \text{tr}(g|\mathfrak{L}_n) = \text{tr}(g|\mathfrak{L}_{(n,0)}) + \text{tr}(g|\mathfrak{L}_{(n,1)}) \\
&= \sum_{\substack{d|n \\ d:\text{odd}}} \frac{1}{d} \mu(d) \sum_{s_0+s_1=n/d} \frac{(|s|-1)!}{s!} \left(\sum_{\substack{\lambda \vdash ds_0 \\ \mu \vdash ds_1}} \chi_\lambda^{(d^{s_0})} \chi_\mu^{(d^{s_1})} S_\lambda(x) S_\mu(y) \right) \\
&+ \sum_{\substack{d|n \\ d:\text{even}}} \frac{1}{d} \mu(d) \sum_{s_0+s_1=n/d} (-1)^{s_1} \frac{(|s|-1)!}{s!} \left(\sum_{\substack{\lambda \vdash ds_0 \\ \mu \vdash ds_1}} \chi_\lambda^{(d^{s_0})} \chi_\mu^{(d^{s_1})} S_\lambda(x) S_\mu(y) \right) \\
&= \sum_{\substack{l(\lambda) \leq k \\ l(\mu) \leq l \\ |\lambda|+|\mu|=n}} a_{\lambda\mu} S_\lambda(x) S_\mu(y),
\end{aligned}$$

where

$$\begin{aligned}
a_{\lambda\mu} &= \sum_{\substack{d|n \\ d:\text{odd}}} \frac{\mu(d)}{d} \sum_{\substack{s_0+s_1=n/d \\ ds_0=|\lambda|}} \frac{(|s|-1)!}{s!} \chi_\lambda^{(d^{s_0})} \chi_\mu^{(d^{s_1})} \\
&+ \sum_{\substack{d|n \\ d:\text{even}}} \frac{\mu(d)}{d} \sum_{\substack{s_0+s_1=n/d \\ ds_0=|\lambda|}} (-1)^{s_1} \frac{(|s|-1)!}{s!} \chi_\lambda^{(d^{s_0})} \chi_\mu^{(d^{s_1})} \\
&= \frac{1}{n} \sum_{d| |\lambda|, |\mu|} \mu(d) \frac{(n/d)!}{(|\lambda|/d)! (|\mu|/d)!} (-1)^{(d-1) \frac{|\mu|}{d}} \chi_\lambda^{(d^{|\lambda|/d})} \chi_\mu^{(d^{|\mu|/d})}.
\end{aligned}$$

Note that $a_{\lambda\mu}$ is the multiplicity of $W_\lambda \otimes W_\mu$ in \mathfrak{L}_n , where W_λ (resp. W_μ) is the irreducible $GL(k)$ -module (resp. $GL(l)$ -module) corresponding to λ (resp. μ). Recall that c_λ is the multiplicity of V_λ in \mathfrak{L}_n . By Proposition 5.2, we have

$$(5.13) \quad \text{tr}(g|\mathfrak{L}_n) = \sum_{\lambda \in H} c_\lambda H S_\lambda(x, y).$$

Therefore, we obtain:

Proposition 5.3 *If $\lambda = (\tau_1, \tau_2, \dots, \tau_k, \tau_{k+1}, \dots)$, then we have*

$$(5.14) \quad c_\lambda = a_{\lambda_0 \lambda_1} - \sum_{\substack{\mu \in H - \{\lambda\} \\ \mu_0 > \lambda_0}} c_\mu N_{\lambda'_0 \lambda_1}^{\mu'},$$

where $\lambda_0 = (\tau_1, \dots, \tau_k)$ and $\lambda_1 = (\tau_{k+1}, \dots)'$.

Proof. Comparing the equations (5.13) and (5.14), we have

$$\begin{aligned}
\sum_{\mu \in H} c_\mu H S_\mu(x, y) &= c_\lambda H S_\lambda(x, y) + \sum_{\mu \neq \lambda} c_\mu H S_\mu(x, y) \\
&= c_\lambda \left(\sum_{\nu < \lambda} S_\nu(x) S_{\lambda' / \nu'}(y) \right) + \sum_{\mu \neq \lambda} c_\mu H S_\mu(x, y) \\
&= c_\lambda (S_{\lambda_0}(x) S_{\lambda_1}(y) + (*)) + \sum_{\mu \neq \lambda} c_\mu \left(\sum_{\tau < \mu} S_\tau(x) \sum_{\sigma} N_{\tau' \sigma}^{\mu'} S_\sigma(y) \right) \\
&= \left(c_\lambda + \sum_{\substack{\mu \in H \\ \mu_0 > \lambda_0}} c_\mu N_{\lambda'_0 \lambda_1}^{\mu'} \right) S_{\lambda_0}(x) S_{\lambda_1}(y) + (*)' \\
&= \sum_{\substack{l(\lambda) \leq k \\ l(\mu) \leq l \\ |\lambda| + |\mu| = n}} a_{\lambda \mu} S_\lambda(x) S_\mu(y).
\end{aligned}$$

Since $\{ S_\lambda(x) S_\mu(y) \mid l(\lambda) \leq k, l(\mu) \leq l \}$ is linearly independent ([8]), we obtain

$$a_{\lambda_0 \lambda_1} = c_\lambda + \sum_{\substack{\mu \in H - \{\lambda\} \\ \mu_0 > \lambda_0}} c_\mu N_{\lambda'_0 \lambda_1}^{\mu'},$$

which completes the proof. \square

If $l(\lambda) \leq k$ (or $|\lambda_0| = n$) for $\lambda \in H(k, l; n)$, then (5.14) yields

$$(5.15) \quad c_\lambda = a_{\lambda_0 \lambda_1} = \frac{1}{n} \sum_{d|n} \mu(d) \chi_\lambda^{(d^{n/d})}.$$

If all c_λ 's are known for $m \leq |\lambda_0| \leq n$, then for any $\lambda \in H(k, l; n)$ with $|\lambda_0| = m - 1$, we have

$$\begin{aligned}
(5.16) \quad c_\lambda &= a_{\lambda_0 \lambda_1} - \sum_{\substack{\mu \in H - \{\lambda\} \\ \mu_0 > \lambda_0}} c_\mu N_{\lambda'_0 \lambda_1}^{\mu'} \\
&= a_{\lambda_0 \lambda_1} - \sum_{\substack{\mu \in H \\ \mu_0 > \lambda_0 \\ |\mu_0| \geq m}} c_\mu N_{\lambda'_0 \lambda_1}^{\mu'},
\end{aligned}$$

since $\mu_0 > \lambda_0$ and $|\mu_0| = m - 1$ imply $\mu_0 = \lambda_0$, and in this case, $N_{\lambda'_0 \lambda_1}^{\mu'} = 0$ unless $\lambda = \mu$ (cf. [46]). Following the above inductive step, we can completely determine the value of all c_λ for $\lambda \in H(k, l; n)$:

Proposition 5.4 *Under the above hypothesis, we have*

$$(5.17) \quad \mathfrak{L}_n(V) = \bigoplus_{\lambda \in H(k, l; n)} V_\lambda^{\oplus c_\lambda},$$

where V_λ is the irreducible $gl(k, l)$ -module corresponding to λ and c_λ is determined by (5.14). \square

Remark. Note that the multiplicity c_λ is given explicitly in closed form when $l(\lambda) \leq k$ and is given by a recursive formula otherwise.

Corollary 5.5 (cf. [3], [23]) *Under the above hypothesis, if $V_1 = 0$, then we have*

$$(5.18) \quad \mathfrak{L}_n(V) = \bigoplus_{\substack{\lambda \vdash n \\ l(\lambda) \leq k}} V_\lambda^{\oplus c_\lambda},$$

$$(5.19) \quad c_\lambda = \frac{1}{n} \sum_{d|n} \mu(d) \chi_\lambda^{(d^{n/d})},$$

where V_λ is the irreducible $gl(k)$ -module corresponding to λ . \square

6 Generalized Kac-Moody Superalgebras

The *generalized Kac-Moody superalgebras* arise naturally in the context of *Monstrous Moonshine* ([5], [6]), automorphic forms with infinite product expansions ([7], [18]–[20]), and the string theory ([22]). In this section, we give a conjectural formula for the homology modules over (the negative part of) generalized Kac-Moody superalgebras, and derive a closed form supertrace formula for generalized Kac-Moody superalgebras with group actions.

Let I be a countable (possibly infinite) index set. A real square matrix $A = (a_{ij})_{i,j \in I}$ is called a *Borcherds-Cartan matrix* if it satisfies: (i) $a_{ii} = 2$ or $a_{ii} \leq 0$ for all $i \in I$, (ii) $a_{ij} \leq 0$ if $i \neq j$, and $a_{ij} \in \mathbb{Z}$ if $a_{ii} = 2$, (iii) $a_{ij} = 0$ implies $a_{ji} = 0$. We say that an index i is *real* if $a_{ii} = 2$ and *imaginary* if $a_{ii} \leq 0$. We denote by $I^{re} = \{i \in I \mid a_{ii} = 2\}$, $I^{im} = \{i \in I \mid a_{ii} \leq 0\}$. Let $\underline{m} = (m_i \in \mathbb{Z}_{>0} \mid i \in I)$ be a sequence of positive integers such that $m_i = 1$ for all $i \in I^{re}$. We call \underline{m} the *charge* of A . In this paper, we assume that the Borcherds-Cartan matrix A is *symmetrizable*, i.e., there is a diagonal matrix $D = \text{diag}(s_i \mid i \in I)$ with $s_i > 0$ ($i \in I$) such that DA is symmetric.

Let $C = (\theta_{ij})_{i,j \in I}$ be a complex matrix satisfying $\theta_{ij}\theta_{ji} = 1$ for all $i, j \in I$. Thus we have $\theta_{ii} = \pm 1$ for all $i \in I$. We call $i \in I$ an *even index* if $\theta_{ii} = 1$ and an *odd index* if $\theta_{ii} = -1$. We denote by I^{even} (resp. I^{odd}) the set of all even (resp. odd) indices. We say that a Borcherds-Cartan matrix $A = (a_{ij})_{i,j \in I}$ is *colored by C* if $a_{ii} = 2$ and $\theta_{ii} = -1$ imply a_{ij} are even integers for all $j \in I$. In this case, the matrix C is called a *coloring matrix* of A .

Let $\mathfrak{h} = (\bigoplus_{i \in I} \mathbb{C}h_i) \oplus (\bigoplus_{i \in I} \mathbb{C}d_i)$ be a complex vector space with a basis $\{h_i, d_i \mid i \in I\}$, and for each $i \in I$ define a linear functional $\alpha_i \in \mathfrak{h}^*$ by

$$(6.1) \quad \alpha_i(h_j) = a_{ji}, \quad \alpha_i(d_j) = \delta_{ij} \quad \text{for all } j \in I.$$

The free abelian group $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ generated by α_i 's ($i \in I$) is called the *root lattice* associated with A . Let $\Pi = \{\alpha_i \mid i \in I\}$ and B be a basis of \mathfrak{h}^* extending Π . Set $B' = B \setminus \Pi$. Since A is assumed to be symmetrizable, there is a symmetric bilinear form (\mid) on \mathfrak{h}^* defined by

$$(6.2) \quad \begin{aligned} (\alpha_i \mid \alpha_j) &= s_i a_{ij} \quad \text{for } i, j \in I, \\ (\lambda \mid \alpha_i) &= \lambda(s_i h_i) \quad \text{for } \lambda \in B', \\ (\lambda \mid \mu) &= 0 \quad \text{for } \lambda, \mu \in B' \end{aligned}$$

(cf. see, for example, [24]). Let $Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ and $Q^- = -Q^+$. There is a partial ordering \geq on Q given by $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q^+$. The coloring matrix $C = (\theta_{ij})_{i,j \in I}$ defines a bimultiplicative map $\theta : Q \times Q \rightarrow \mathbb{C}^\times$ on Q by

$$(6.3) \quad \begin{aligned} \theta(\alpha_i, \alpha_j) &= \theta_{ij} \quad \text{for all } i, j \in I, \\ \theta(\alpha + \beta, \gamma) &= \theta(\alpha, \gamma)\theta(\beta, \gamma), \\ \theta(\alpha, \beta + \gamma) &= \theta(\alpha, \beta)\theta(\alpha, \gamma) \end{aligned}$$

for all $\alpha, \beta, \gamma \in Q$. Note that, since $\theta_{ij}\theta_{ji} = 1$ for all $i, j \in I$, θ satisfies

$$(6.4) \quad \theta(\alpha, \beta)\theta(\beta, \alpha) = 1 \quad \text{for all } \alpha, \beta \in Q.$$

That is, θ is a coloring map on Q . In particular, $\theta(\alpha, \alpha) = \pm 1$ for all $\alpha \in Q$. We say $\alpha \in Q$ is *even* if $\psi(\alpha) = \theta(\alpha, \alpha) = 1$ and *odd* if $\psi(\alpha) = \theta(\alpha, \alpha) = -1$.

The *generalized Kac-Moody superalgebra* $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, C)$ associated with a symmetrizable Borcherds-Cartan matrix $A = (a_{ij})_{i,j \in I}$ of charge $\underline{m} = (m_i \mid i \in I)$ with a coloring matrix $C = (\theta_{ij})_{i,j \in I}$ is the θ -colored Lie superalgebra generated by the elements h_i, d_i ($i \in I$), e_{ik}, f_{ik} ($i \in I, k = 1, 2, \dots, m_i$) with the defining relations:

$$(6.5) \quad \begin{aligned} [h_i, h_j] &= [h_i, d_j] = [d_i, d_j] = 0, \\ [h_i, e_{jl}] &= a_{ij}e_{jl}, \quad [h_i, f_{jl}] = -a_{ij}f_{jl}, \\ [d_i, e_{jl}] &= \delta_{ij}e_{jl}, \quad [d_i, f_{jl}] = -\delta_{ij}f_{jl}, \\ [e_{ik}, f_{jl}] &= \delta_{ij}\delta_{kl}h_i, \\ (ade_{ik})^{1-a_{ij}}(e_{jl}) &= (adf_{ik})^{1-a_{ij}}(f_{jl}) = 0 \quad \text{if } a_{ii} = 2 \text{ and } i \neq j, \\ [e_{ik}, e_{jl}] &= [f_{ik}, f_{jl}] = 0 \quad \text{if } a_{ij} = 0 \end{aligned}$$

for $i, j \in I, k = 1, \dots, m_i, l = 1, \dots, m_j$.

The abelian subalgebra $\mathfrak{h} = (\bigoplus_{i \in I} \mathbb{C}h_i) \oplus (\bigoplus_{i \in I} \mathbb{C}d_i)$ is called the *Cartan subalgebra* of \mathfrak{g} , and the linear functionals $\alpha_i \in \mathfrak{h}^*$ ($i \in I$) defined by (6.1) are called the *simple roots* of \mathfrak{g} . For each $i \in I^e$, let $r_i \in GL(\mathfrak{h}^*)$ be the *simple reflection* on \mathfrak{h}^* defined by

$$r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i \quad \text{for } \lambda \in \mathfrak{h}^*.$$

The subgroup W of $GL(\mathfrak{h}^*)$ generated by the r_i 's ($i \in I^{re}$) is called the *Weyl group* of the generalized Kac-Moody superalgebra \mathfrak{g} .

The generalized Kac-Moody superalgebra $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, C)$ has the *root space decomposition* $\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$, where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

Note that $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_{i,1} \oplus \cdots \oplus \mathbb{C}e_{i,m_i}$ and $\mathfrak{g}_{-\alpha_i} = \mathbb{C}f_{i,1} \oplus \cdots \oplus \mathbb{C}f_{i,m_i}$. We say that $\alpha \in Q^\times$ is a *root* of \mathfrak{g} if $\mathfrak{g}_\alpha \neq 0$. The subspace \mathfrak{g}_α is called the *root space* of \mathfrak{g} attached to α . A root α is called *real* if $(\alpha|\alpha) > 0$ and *imaginary* if $(\alpha|\alpha) \leq 0$. In particular, a simple root α_i is real if $a_{ii} = 2$ (i.e., $i \in I^{re}$) and imaginary if $a_{ii} \leq 0$ (i.e., $i \in I^{im}$). Note that the imaginary simple roots may have multiplicity > 1 . A root $\alpha > 0$ (resp. $\alpha < 0$) is called *positive* (resp. *negative*). One can show that all the roots are either positive or negative. We denote by Φ , Φ^+ , and Φ^- the set of all roots, positive roots, and negative roots, respectively. We also denote by Φ_0 (resp. Φ_1) the set of all even (resp. odd) roots of \mathfrak{g} . Hence, for example, Φ_0^+ will denote the set of all positive even roots of \mathfrak{g} . Define the subalgebras $\mathfrak{g}^\pm = \bigoplus_{\alpha \in \Phi^\pm} \mathfrak{g}_\alpha$. Then we have the *triangular decomposition*:

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+.$$

We can define a nondegenerate symmetric bilinear form on \mathfrak{h} by

$$(6.6) \quad (h_i|h) = \frac{1}{s_i}\alpha_i(h) \quad \text{and} \quad (d_i|d_j) = 0$$

for all $h \in \mathfrak{h}$, $i, j \in I$ (cf. [31]). The symmetric bilinear form (\mid) on \mathfrak{h} defined by (6.6) can be extended to a nondegenerate, *supersymmetric*, and *invariant* bilinear form on the generalized Kac-Moody superalgebra \mathfrak{g} :

Proposition 6.1 (cf. [24], [30]) *Let $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, C)$ be the generalized Kac-Moody superalgebra associated with a Borcherds-Cartan data (A, \underline{m}, C) . Then there exists a nondegenerate bilinear form (\mid) on \mathfrak{g} satisfying the following conditions:*

(a) *The bilinear form (\mid) is supersymmetric, i.e., we have*

$$(x|y) = \theta(\beta, \alpha)(y|x) \quad \text{for } x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta.$$

(b) *The bilinear form (\mid) is invariant, i.e., we have*

$$([x, y]|z) = (x|[y, z]) \quad \text{for } x, y, z \in \mathfrak{g}.$$

(c) *The bilinear form $(\mid)|_{\mathfrak{h}}$ is given by (6.6).*

(d) *$(\mathfrak{g}_\alpha|\mathfrak{g}_\beta) = 0$ if $\alpha + \beta \neq 0$.*

(e) *The root spaces \mathfrak{g}_α and $\mathfrak{g}_{-\alpha}$ are nondegenerately paired with respect to (\mid) , i.e., the bilinear form $(\mid)|_{\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}}$ is nondegenerate.*

(f) *If $\phi : Q \rightarrow \mathfrak{h}$ is a linear map satisfying $\phi(\alpha_i) = s_i h_i$ for all $i \in I$, then we have*

$$[x, y] = \theta(\alpha, \alpha)(x|y)\phi(\alpha) \quad \text{for } x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}. \quad \square$$

A \mathfrak{g} -module V is called \mathfrak{h} -diagonalizable if it admits a *weight space decomposition* $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$, where

$$V_\mu = \{v \in V \mid h \cdot v = \mu(h)v \text{ for all } h \in \mathfrak{h}\}.$$

If $V_\mu \neq 0$, then μ is called a *weight* of V and V_μ is called the μ -*weight space*. We denote by $P(V)$ the set of all weights of V . When all the weight spaces are finite dimensional, we define the *character* of V to be

$$(6.7) \quad \text{ch} V = \sum_{\mu \in \mathfrak{h}^*} (\dim V_\mu) e^\mu,$$

where e^μ are the basis elements of the group algebra $\mathbb{C}[\mathfrak{h}^*]$ with the multiplication given by $e^\mu e^\nu = e^{\mu+\nu}$ for $\mu, \nu \in \mathfrak{h}^*$.

We denote by \mathcal{O} the category of \mathfrak{h} -diagonalizable \mathfrak{g} -modules with finite dimensional weight spaces such that there exist a finite number of linear functionals $\lambda_1, \dots, \lambda_s$ satisfying $P(V) \subset \bigcup_{i=1}^s D(\lambda_i)$, where $D(\lambda) = \{\mu \in \mathfrak{h}^* \mid \mu \leq \lambda\}$. The morphisms in \mathcal{O} are the usual \mathfrak{g} -module homomorphisms.

The most important example of the \mathfrak{g} -modules in category \mathcal{O} may be the class of highest weight modules. An \mathfrak{h} -diagonalizable \mathfrak{g} -module V is called a *highest weight module* with highest weight $\lambda \in \mathfrak{h}^*$ if there is a nonzero vector $v_\lambda \in V$ such that (i) $e_{ik} \cdot v_\lambda = 0$ for all $i \in I$, $k = 1, \dots, m_i$, (ii) $h \cdot v_\lambda = \lambda(h)v_\lambda$ for all $h \in \mathfrak{h}$, (iii) $V = U(\mathfrak{g}) \cdot v_\lambda$. The vector v_λ is called a *highest weight vector*. For a highest weight module V with highest weight λ , we have (i) $V = U(\mathfrak{g}^-) \cdot v_\lambda$, (ii) $V = \bigoplus_{\mu \leq \lambda} V_\mu$, $V_\lambda = \mathbb{C}v_\lambda$, and (iii) $\dim V_\mu < \infty$ for all $\mu \leq \lambda$.

Let $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{g}^+$, and let \mathbb{C}_λ be the 1-dimensional \mathfrak{b}^+ -module defined by $h \cdot 1 = \lambda(h)1$ for all $h \in \mathfrak{h}$ and $\mathfrak{g}^+ \cdot 1 = 0$. The induced module $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} \mathbb{C}_\lambda$ is called the *Verma module* over \mathfrak{g} with highest weight λ . Every highest weight \mathfrak{g} -module with highest weight λ is a homomorphic image of $M(\lambda)$ and the Verma module $M(\lambda)$ contains a unique maximal submodule $J(\lambda)$. Hence the quotient $V(\lambda) = M(\lambda)/J(\lambda)$ is irreducible.

Let V be a \mathfrak{g} -module in category \mathcal{O} . We define the *Casimir operator* on V as follows (cf. [24], [30]). Take a linear functional $\rho \in \mathfrak{h}^*$ satisfying $\rho(h_i) = \frac{1}{2}a_{ii}$ for all $i \in I$. Such a linear functional is called a *Weyl vector* of \mathfrak{g} . Note that $(\rho|\alpha_i) = \rho(s_i h_i) = \frac{1}{2}(\alpha_i|\alpha_i)$ ($i \in I$). For each positive root $\alpha \in \Phi^+$, choose a basis $\{e_\alpha^{(i)} \mid i = 1, \dots, \dim \mathfrak{g}_\alpha\}$ of the root space \mathfrak{g}_α and let $\{e_{-\alpha}^{(i)} \mid i = 1, \dots, \dim \mathfrak{g}_\alpha\}$ be the *dual basis* of $\mathfrak{g}_{-\alpha}$ in the sense that $(e_\alpha^{(i)}|e_{-\alpha}^{(j)}) = \delta_{ij}\theta(\alpha, \alpha)$. Then for a weight vector $v \in V_\lambda$ of weight λ , the Casimir operator Ω on V is defined by

$$(6.8) \quad \Omega(v) = (\lambda + 2\rho|\lambda)v_\lambda + 2 \sum_{\alpha \in \Phi^+} \sum_{i=1}^{\dim \mathfrak{g}_\alpha} e_{-\alpha}^{(i)} e_\alpha^{(i)} v_\lambda.$$

The Casimir operator commutes with the action of \mathfrak{g} on V . Recall that a weight vector $v \in V$ is called a *primitive vector* if there is a \mathfrak{g} -submodule W of V such that $v \notin W$

and $e_{ik} \cdot v \in W$ for all $i \in I$, $k = 1, 2, \dots, m_i$ (cf. [31]). The Casimir operator satisfies the following properties:

Proposition 6.2 (a) *If V is a highest weight \mathfrak{g} -module in Category \mathcal{O} with highest weight Λ , then we have*

$$(6.9) \quad \Omega = (\Lambda + 2\rho|\Lambda)I_V.$$

(b) *If v is a primitive vector with weight λ , then there exists a \mathfrak{g} -submodule $W \subset V$ such that $v \notin W$ and*

$$(6.10) \quad \Omega(v) = (\lambda + 2\rho|\lambda)v \mod W. \quad \square$$

Let P^+ be the set of all linear functionals $\lambda \in \mathfrak{h}^*$ satisfying

$$(6.11) \quad \begin{cases} \lambda(h_i) \in \mathbb{Z}_{\geq 0} & \text{for all } i \in I^{re}, \\ \lambda(h_i) \in 2\mathbb{Z}_{\geq 0} & \text{for all } i \in I^{re} \cap I^{odd}, \\ \lambda(h_i) \geq 0 & \text{for all } i \in I^{im}. \end{cases}$$

The elements of P^+ are called the *dominant integral weights*. For a dominant integral weight $\Lambda \in P^+$, let

$$(6.12) \quad \begin{aligned} \Phi^+(\Lambda) &= \{\beta = \sum_{i \in I^{im}} k_i \alpha_i \in Q^+ \mid (\Lambda|\alpha_i) = 0 \text{ for } k_i \geq 1, \\ &\quad (\alpha_i|\alpha_j) = 0 \text{ for } k_i, k_j \geq 1, i \neq j, (\alpha_i|\alpha_i) = 0 \text{ for } k_i \geq 2\}. \end{aligned}$$

For such an element $\beta \in \Phi^+(\Lambda)$, we denote $|\beta| = \sum_{i \in I^{im}} k_i$ and

$$(6.13) \quad \varepsilon(\beta) = \prod_{i \in I^{im} \cap I^{even}} \binom{m_i}{k_i} \prod_{j \in I^{im} \cap I^{odd}} \binom{m_j + k_j - 1}{k_j}.$$

Then the character of the irreducible highest weight module $V(\Lambda)$ with highest weight $\Lambda \in P^+$ is determined by the *Weyl-Kac-Borcherds formula*:

Proposition 6.3 ([47], [51])

$$(6.14) \quad chV(\Lambda) = \frac{\prod_{\alpha \in \Phi_1^-} (1 + e^\alpha)^{dim \mathfrak{g}_\alpha}}{\prod_{\alpha \in \Phi_0^-} (1 - e^\alpha)^{dim \mathfrak{g}_\alpha}} \sum_{\substack{w \in W \\ \beta \in \Phi^+(\Lambda)}} (-1)^{l(w) + |\beta|} \varepsilon(\beta) e^{w(\Lambda + \rho - \beta) - \rho}.$$

Letting $\Lambda = 0$, we obtain the denominator identity:

$$(6.15) \quad \frac{\prod_{\alpha \in \Phi_0^-} (1 - e^\alpha)^{dim \mathfrak{g}_\alpha}}{\prod_{\alpha \in \Phi_1^-} (1 + e^\alpha)^{dim \mathfrak{g}_\alpha}} = \sum_{\substack{w \in W \\ \beta \in \Phi^+(0)}} (-1)^{l(w) + |\beta|} \varepsilon(\beta) e^{w(\rho - \beta) - \rho}. \quad \square$$

Corollary 6.4 *A highest weight \mathfrak{g} -module V with highest weight $\Lambda \in P^+$ and highest weight vector v_Λ is irreducible if and only if it satisfies:*

$$(6.16) \quad \begin{aligned} f_{ik}^{\Lambda(h_i)+1} \cdot v_\Lambda &= 0 \quad \text{if } i \in I^{re}, \\ f_{ik} \cdot v_\Lambda &= 0 \quad (k = 1, \dots, m_i) \quad \text{if } \Lambda(h_i) = 0. \end{aligned} \quad \square$$

Suppose a group G acts on the generalized Kac-Moody superalgebra $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, C)$ by Lie superalgebra automorphisms of \mathfrak{g}_\pm preserving the root space decomposition. Assume further that $\text{str}(g|\mathfrak{g}_\alpha) = \text{str}(g|\mathfrak{g}_{-\alpha})$ for all $g \in G$ and $\alpha \in Q^+$. We would like to apply our supertrace formula (2.17) to derive a closed form formula for $\text{str}(g|\mathfrak{g}_\alpha)$ ($g \in G, \alpha \in Q$).

Let J be a finite subset of I^{re} , and let $\Phi_J = \Phi \cap (\sum_{j \in J} \mathbb{Z}\alpha_j)$, $\Phi_J^\pm = \Phi^\pm \cap \Phi_J$, and $\Phi^\pm(J) = \Phi^\pm \setminus \Phi_J^\pm$. We also denote $Q_J = Q \cap (\sum_{j \in J} \mathbb{Z}\alpha_j)$, $Q_J^\pm = Q^\pm \cap Q_J$, and $Q^\pm(J) = Q^\pm \setminus Q_J^\pm$. Let $\mathfrak{g}_0^{(J)} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Phi_J} \mathfrak{g}_\alpha)$, and $\mathfrak{g}_\pm^{(J)} = \bigoplus_{\alpha \in \Phi^\pm(J)} \mathfrak{g}_\alpha$. Then we have the *triangular decomposition*:

$$(6.17) \quad \mathfrak{g} = \mathfrak{g}_-^{(J)} \oplus \mathfrak{g}_0^{(J)} \oplus \mathfrak{g}_+^{(J)},$$

where $\mathfrak{g}_0^{(J)}$ is the Kac-Moody superalgebra (with an extended Cartan subalgebra) associated with the generalized Cartan matrix $A_J = (a_{ij})_{i,j \in J}$ and the set of odd indices $J^{odd} = J \cap I^{odd} = \{j \in J \mid \theta_{jj} = -1\}$, and $\mathfrak{g}_-^{(J)}$ (resp. $\mathfrak{g}_+^{(J)}$) is a direct sum of irreducible highest weight (resp. lowest weight) modules over $\mathfrak{g}_0^{(J)}$ (cf. [30]).

To apply our supertrace formula (2.17) to the Lie superalgebra $\mathfrak{g}_-^{(J)}$, we need to compute the generalized supercharacters of the homology modules $H_k(\mathfrak{g}_-^{(J)})$. We conjecture that the $\mathfrak{g}_0^{(J)}$ -module structure of $H_k(\mathfrak{g}_-^{(J)})$ is determined by *Kostant's formula*. More precisely, let W_J be the subgroup of W generated by the simple reflections r_j with $j \in J$, and let $W(J) = \{w \in W \mid \Phi_w \subset \Phi^+(J)\}$, where $\Phi_w = \{\alpha \in \Phi^+ \mid w^{-1}\alpha < 0\}$. Thus W_J is the Weyl group of the Kac-Moody superalgebra $\mathfrak{g}_0^{(J)}$ and $W(J)$ is the set of right coset representatives of W_J in W . That is, $W = W_J W(J)$. Let us denote by $\Phi_{J,i}^\pm = \Phi_J \cap \Phi_i^\pm$ ($i = 0, 1$) and $\Phi_i^\pm(J) = \Phi_i^\pm \setminus \Phi_{J,i}^\pm$ ($i = 0, 1$). Then we have the following conjecture for the structure of $H_k(\mathfrak{g}_-^{(J)})$:

Conjecture:

$$(6.18) \quad H_k(\mathfrak{g}_-^{(J)}) \cong \sum_{\substack{w \in W(J) \\ \beta \in \Phi^+(0) \\ l(w) + |\beta| = k}} V_J(w(\rho - \beta) - \rho)^{\oplus \varepsilon(\beta)},$$

where $V_J(\mu)$ denotes the irreducible highest weight module over the Kac-Moody superalgebra $\mathfrak{g}_0^{(J)}$ with highest weight μ . \square

Remark. The formula (6.18) was first introduced by Kostant for finite dimensional simple Lie algebras ([43]). It was generalized to symmetrizable Kac-Moody algebras

([17], [44]) and generalized Kac-Moody algebras ([49]). But, in general, it is not true for Lie superalgebras. Still, we conjecture that Kostant's formula holds for generalized Kac-Moody superalgebras associated with the Borchers-Cartan matrix which is colored by a coloring matrix.

For each $k \geq 1$, let

$$(6.19) \quad H_k^{(J)} = \sum_{\substack{w \in W(J) \\ \beta \in \Phi^+(0) \\ l(w) + |\beta| = k}} V_J(w(\rho - \beta) - \rho)^{\oplus \varepsilon(\beta)},$$

and define the *homology superspace*

$$(6.20) \quad H^{(J)} = \sum_{k=1}^{\infty} (-1)^{k+1} H_k^{(J)} = H_1^{(J)} \ominus H_2^{(J)} \oplus H_3^{(J)} \ominus \cdots,$$

an alternating direct sum of superspaces. Then, for all $g \in G$ and $\alpha \in Q_-$, we have

$$(6.21) \quad \text{str}(g|H_\alpha^{(J)}) = \sum_{\substack{w \in W(J) \\ \beta \in \Phi^+(0) \\ l(w) + |\beta| \geq 1}} (-1)^{l(w) + |\beta| + 1} \varepsilon(\beta) \text{str}(g|V_J(w(\rho - \beta) - \rho)_\alpha),$$

and the generalized denominator identity can be written as

$$(6.22) \quad \prod_{\alpha \in \Phi^-(J)} \exp \left(- \sum_{k=1}^{\infty} \frac{1}{k} \text{str}(g^k | \mathfrak{g}_\alpha) E^{k\alpha} \right) = 1 - \text{sch}_g H^{(J)} \\ = \sum_{\substack{w \in W(J) \\ \beta \in \Phi^+(0)}} (-1)^{l(w) + |\beta|} \varepsilon(\beta) \text{sch}_g V_J(w(\rho - \beta) - \rho).$$

Let $P(H^{(J)}) = \{\tau \in Q_- \mid \text{str}(g|H_\tau^{(J)}) \neq 0\} = \{\tau_1, \tau_2, \tau_3, \dots\}$, and $t_g(i) = \text{str}(g|H_{\tau_i}^{(J)})$. For $\tau \in Q_-$, define the set $T^{(J)}(\tau)$ of all partitions of τ into a sum of τ_i 's as in (2.15) and define the Witt partition function $W_g^{(J)}(\tau)$ as in (2.16). Then, our supertrace formula (2.17) yields:

Theorem 6.5 *Suppose that a group G acts on the generalized Kac-Moody superalgebra $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, C)$ by Lie superalgebra automorphisms of \mathfrak{g}_\pm preserving the root space decomposition. Assume further that $\text{str}(g|\mathfrak{g}_\alpha) = \text{str}(g|\mathfrak{g}_{-\alpha})$ for all $g \in G$ and $\alpha \in Q^+$. Let J be a finite subset of I^{re} . Then, for all $g \in G$ and $\alpha \in \Phi^-(J)$, we have*

$$(6.23) \quad \text{str}(g|\mathfrak{g}_\alpha) = \sum_{d|\alpha} \frac{1}{d} \mu(d) W_{g^d}^{(J)} \left(\frac{\alpha}{d} \right) \\ = \sum_{d|\alpha} \frac{1}{d} \mu(d) \sum_{s \in T^{(J)}(\alpha/d)} \frac{(|s| - 1)!}{s!} \prod t_{g^d}(i)^{s_i},$$

where μ is the classical Möbius function, and, for a positive integer d , $d|\alpha$ denotes $\alpha = d\tau$ for some $\tau \in Q_-$, in which case $\alpha/d = \tau$. \square

We would like to remind the readers that our supertrace formula (6.23) has been derived under the assumption that Kostant's formula holds for generalized Kac-Moody superalgebras. However, in some special cases, we can derive the supertrace formula for generalized Kac-Moody superalgebras without assuming Kostant's formula.

For example, suppose that the Borcherds-Cartan matrix A satisfies: (i) the set I^{re} is finite, (ii) $a_{ij} \neq 0$ for all $i, j \in I^{im}$. Then it is known that the Lie superalgebra $\mathfrak{g}_-^{(J)} = \bigoplus_{\alpha \in \Delta^-(J)} \mathfrak{g}_\alpha$ (resp. $\mathfrak{g}_+^{(J)} = \bigoplus_{\alpha \in \Delta^+(J)} \mathfrak{g}_\alpha$) is isomorphic to the free Lie superalgebra generated by the superspace $V = \bigoplus_{j \in I^{im}} V_J(-\alpha_j)^{\oplus m_j}$ (resp. $V^* = \bigoplus_{j \in I^{im}} V_J^*(-\alpha_j)^{\oplus m_j}$), where $V_J(\mu)$ (resp. $V_J^*(\mu)$) denotes the irreducible highest weight (resp. lowest weight) module over the Kac-Moody superalgebra $\mathfrak{g}_0^{(J)}$ with highest weight μ (resp. lowest weight $-\mu$) (cf. [40]). Hence, for any $g \in G$, the generalized denominator identity of the Lie superalgebra $\mathfrak{g}_-^{(J)}$ is the same as

$$(6.24) \quad \prod_{\alpha \in \Phi^-(J)} \exp \left(- \sum_{k=1}^{\infty} \frac{1}{k} \text{str}(g^k | \mathfrak{g}_\alpha) E^{k\alpha} \right) = 1 - \text{sch}_g V \\ = 1 - \sum_{j \in I^{im}} m_j \text{sch}_g V_J(-\alpha_j).$$

Hence, by our supertrace formula (2.17), we obtain

$$(6.25) \quad \text{str}(g | \mathfrak{g}_\alpha) = \sum_{d|\alpha} \frac{1}{d} \mu(d) \sum_{s \in T^{(J)}(\alpha/d)} \frac{(|s| - 1)!}{s!} \prod t_{g^d(i)}^{s_i},$$

where $t_g(i) = \text{str}(g | V_{\tau_i}) = \sum_{j \in I^{im}} m_j \text{str}(g | V_J(-\alpha_j)_{\tau_i})$ for $i = 1, 2, 3, \dots$.

7 Monstrous Lie Superalgebras

Let $F(q) = \sum_{n=-1}^{\infty} f(n)q^n = q^{-1} + f(1)q + f(2)q^2 + \dots$ be a normalized q -series such that $f(-1) = 1$, $f(0) = 0$, and $f(n) \in \mathbb{Z}$ for all $n \geq 1$. We say that $F(q)$ is *convergent* if it converges for all $q = e^{2\pi i \tau}$, $\text{Im} \tau > 0$. Observe that, for each $m \geq 1$, there exists a unique polynomial $P_m(t) \in \mathbb{Z}[t]$ such that

$$P_m(F) \equiv q^{-m} \pmod{q\mathbb{Z}[[q]]}.$$

A convergent normalized q -series $F(q) = \sum_{n=-1}^{\infty} f(n)q^n = q^{-1} + f(1)q + f(2)q^2 + \dots$ is called *replicable* if for all $m > 0$ and for all $a|m$, there exist normalized q -series

$F^{(a)}(q) = \sum_{n=-1}^{\infty} f^{(a)}(n)q^n = q^{-1} + f^{(a)}(1)q + f^{(a)}(2)q^2 + \cdots$ with $f^{(a)}(-1) = 1$, $f^{(a)}(0) = 0$, and $f^{(a)}(n) \in \mathbb{Z}$ for all $n \geq 1$ such that

$$(7.1) \quad F^{(1)} = F, \quad \text{and} \quad \sum_{\substack{ad=m \\ 0 \leq b < d}} F^{(a)}\left(\frac{a\tau + b}{d}\right) = P_m(F),$$

where $q = e^{2\pi i\tau}$, $\text{Im}\tau > 0$.

In [40], the replicable functions were characterized in terms of product identities. More precisely, a normalized q -series $F(q) = q^{-1} + f(1)q + f(2)q^2 + \cdots = \sum_{n=-1}^{\infty} f(n)q^n$ is replicable if and only if for all $k \geq 1$, there exist normalized q -series $F^{(k)}(q) = \sum_{n=-1}^{\infty} f^{(k)}(n)q^n$ satisfying the product identity

$$(7.2) \quad \prod_{m,n=1}^{\infty} \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k} f^{(k)}(mn) p^{km} q^{kn}\right) = 1 - \sum_{i,j=1}^{\infty} f(i+j-1) p^i q^j.$$

Now, we will consider the generalized Kac-Moody superalgebras associated with the normalized q -series. We take $I = \{-1\} \cup \{1, 2, 3, \dots\}$ to be the index set, and let $A = (-i+j)_{i,j \in I}$ be the Borcherds-Cartan matrix of the Monster Lie algebra ([6]). Consider a normalized q -series $F(q) = \sum_{n=-1}^{\infty} f(n)q^n$ such that $f(-1) = 1$, $f(0) = 0$, and $f(n) \in \mathbb{Z}$ for all $n \geq 1$. We define the charge of the matrix A to be $\underline{m} = (|f(i)| : i \in I)$, and choose a coloring matrix $C = (\theta_{ij})_{i,j \in I}$ such that $\theta_{ii} = 1$ if $f(i) > 0$ and $\theta_{ii} = -1$ if $f(i) < 0$. That is, an index $i \in I$ is even if $f(i) > 0$ and odd if $f(i) < 0$.

The generalized Kac-Moody superalgebra $\mathfrak{L}(F) = \mathfrak{g}(A, \underline{m}, C)$ associated with the above data is called the *Monstrous Lie superalgebra* associated with the normalized q -series $F(q) = \sum_{n=-1}^{\infty} f(n)q^n$. For example, the Monstrous Lie superalgebra associated with the elliptic modular function $J(q) = j(q) - 744$ is the Monster Lie algebra, and the Monstrous Lie superalgebras associated with the Thompson series $T_g(q) = \sum_{n=-1}^{\infty} c_g(n)q^n$ are the Monstrous Lie superalgebras given in [6].

Let $\mathfrak{L}(F)$ be the Monstrous Lie superalgebra associated with a normalized q -series $F(q) = \sum_{n=-1}^{\infty} f(n)q^n$. By our choice of the charge and the coloring matrix, we see that α_{-1} is the only real even simple root, α_i ($i \geq 1$) is an imaginary even simple root with multiplicity $f(i)$ if $f(i) > 0$, and an imaginary odd simple root with multiplicity $-f(i)$ if $f(i) < 0$. If we identify the simple roots α_{-1} with $(1, -1) \in II_{1,1}$ and α_i with $(1, i) \in II_{1,1}$ ($i \geq 1$), then the Monstrous Lie superalgebra $\mathfrak{L}(F)$ becomes a $II_{1,1}$ -graded Lie superalgebra, and we have

$$\mathfrak{L}(F)_+ = \bigoplus_{\substack{m>0 \\ n \in \mathbb{Z}}} \mathfrak{L}(F)_{(m,n)}, \quad \mathfrak{L}(F)_- = \bigoplus_{\substack{m>0 \\ n \in \mathbb{Z}}} \mathfrak{L}(F)_{(-m,n)}.$$

Suppose that a group G acts on $\mathfrak{L}(F)$ by Lie superalgebra automorphisms of $\mathfrak{L}(F)_{\pm}$ preserving the $II_{1,1}$ -gradation. Assume further that $\text{str}(g|\mathfrak{L}(F)_{(m,n)}) = \text{str}(g|\mathfrak{L}(F)_{(-m,-n)})$

for all $g \in G$, $m, n > 0$, and

$$g \cdot f_{-1} = f_{-1}, \quad g \cdot e_{-1} = e_{-1} \quad \text{for all } g \in G.$$

We will apply our supertrace formula (6.23) (actually (6.25)) to this setting.

Let $V_{-1} = \mathfrak{L}_{(-1,1)} = \mathbb{C}f_{-1}$, $V_i = \mathfrak{L}(F)_{(-1,-i)} = \mathbb{C}f_{i,1} \oplus \cdots \oplus \mathbb{C}f_{i,|f(i)|}$ for $i \geq 1$, and $V = \bigoplus_{n=-1}^{\infty} V_n$. For $g \in G$ and $i \in I = \{-1\} \cup \{1, 2, 3, \dots\}$, define $f_g(i) = \text{str}(g|V_i)$. Then $V = \bigoplus_{n=-1}^{\infty} V_n$ is a \mathbb{Z} -graded representation of the group G such that $\text{sch}(V) = F(q) = \sum_{n=-1}^{\infty} f(n)q^n$ and $\text{sch}_g(V) = F_g(q) = \sum_{n=-1}^{\infty} f_g(n)q^n$.

Consider a basis of V_i consisting of the eigenvectors $v_{i,k}$ of $g \in G$ with eigenvalues $\lambda_{i,k}$ ($k = 1, \dots, |f(i)|$). Then we have

$$f_g(i) = \text{str}(g|V_i) = \psi(1, i) \sum_{k=1}^{|f(i)|} \lambda_{i,k}.$$

To apply our supertrace formula (6.23), take $J = \{-1\}$ and consider the corresponding triangular decomposition

$$\mathfrak{L}(F) = \mathfrak{L}(F)_-^{(J)} \oplus \mathfrak{L}(F)_0^{(J)} \oplus \mathfrak{L}(F)_+^{(J)}.$$

Then $\mathfrak{L}(F)_0^{(J)} = \langle e_{-1}, f_{-1}, h_{-1} \rangle + \mathfrak{h} \cong \mathfrak{sl}(2, \mathbb{C}) + \mathfrak{h}$, and the subalgebra $\mathfrak{L}(F)_-^{(J)}$ (resp. $\mathfrak{L}(F)_+^{(J)}$) is isomorphic to the free Lie superalgebra generated by the superspace $H^{(J)} = \bigoplus_{i=1}^{\infty} V_J(-\alpha_i)^{\oplus |f(i)|}$ (resp. $H^{(J)*} = \bigoplus_{i=1}^{\infty} V_J^*(-\alpha_i)^{\oplus |f(i)|}$), where $V_J(-\alpha_i)$ and $V_J^*(-\alpha_i)$ are the i -dimensional irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$ (since $-\alpha_i(h_{-1}) = i - 1$). We will concentrate on the Lie superalgebra $\mathfrak{L}(F)_-^{(J)}$.

Since the weights of $V_J(-\alpha_i)$ are $-\alpha_i = (-1, -i)$, $-\alpha_i - \alpha_{-1} = (-2, -i + 1)$, \dots , $-\alpha_i - (i - 1)\alpha_{-1} = (-i, -1)$, we have

$$P(H^{(J)}) = \{(-i, -j) \mid i, j \in \mathbb{Z}_{>0}\}.$$

Note that each $f_{i,k}$ generates an i -dimensional irreducible representation of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ generated by e_{-1}, f_{-1}, h_{-1} , and hence so does each $v_{i,k}$ ($i \geq 1, k = 1, 2, \dots, |f(i)|$). Thus we have

$$\begin{aligned} H_{(-i,-j)}^{(J)} &= \bigoplus_{k=1}^{|f(i+j-1)|} \mathbb{C}(\text{adf}_{-1})^{i-1}(f_{i+j-1,k}) \\ &= \bigoplus_{k=1}^{|f(i+j-1)|} \mathbb{C}(\text{adf}_{-1})^{i-1}(v_{i+j-1,k}). \end{aligned}$$

It follows that

$$\begin{aligned} g \cdot (\text{adf}_{-1})^{i-1}(v_{i+j-1,k}) &= (\text{ad}(g \cdot f_{-1}))^{i-1}(g \cdot v_{i+j-1,k}) \\ &= \lambda_{i+j-1,k}(\text{adf}_{-1})^{i-1}(v_{i+j-1,k}), \end{aligned}$$

which yields

$$\text{str}(g|H_{(-i,-j)}^{(J)}) = \psi(1, i+j-1) \sum_{k=1}^{|f(i+j-1)|} \lambda_{i+j-1,k} = f_g(i+j-1)$$

for all $g \in G$, $i, j \in \mathbb{Z}_{>0}$. Hence the generalized denominator identity for $g \in G$ of the Lie superalgebra $\mathfrak{L}(F)_-^{(J)}$ is equal to

$$(7.3) \quad \prod_{m,n=1}^{\infty} \exp \left(- \sum_{k=1}^{\infty} \frac{1}{k} \text{str}(g^k | \mathfrak{L}(F)_{(-m,-n)}^{(J)}) p^{km} q^{kn} \right) \\ = 1 - \sum_{i,j=1}^{\infty} f_g(i+j-1) p^i q^j.$$

For $k, l > 0$, we have

$$T^{(J)}(k, l) = T(k, l) = \{s = (s_{ij})_{i,j \geq 1} \mid s_{ij} \in \mathbf{Z}_{\geq 0}, \sum s_{ij}(i, j) = (k, l)\},$$

the set of all partitions of (k, l) into a sum of ordered pairs of positive integers, and the Witt partition function $W_g^{(J)}(k, l)$ is given by

$$W_g^{(J)}(k, l) = \sum_{s \in T(k, l)} \frac{(|s| - 1)!}{s!} \prod \text{str}(g|H_{(-i,-j)}^{(J)})^{s_{ij}} \\ = \sum_{s \in T(k, l)} \frac{(|s| - 1)!}{s!} \prod f_g(i+j-1)^{s_{ij}}.$$

Therefore, our supertrace formula (6.23) yields:

Proposition 7.1 *Let $\mathfrak{L}(F) = \bigoplus_{(m,n) \in II_{1,1}} \mathfrak{L}(F)_{(m,n)}$ be the Monstrous Lie superalgebra associated with a normalized q -series $F(q) = \sum_{n=-1}^{\infty} f(n)q^n$ and let $V = \bigoplus_{n=-1}^{\infty} V_n = \bigoplus_{n=-1}^{\infty} \mathfrak{L}(F)_{(-1,-n)}$. Suppose that a group G acts on $\mathfrak{L}(F)$ by Lie superalgebra automorphisms of $\mathfrak{L}(F)_{\pm}$ preserving the $II_{1,1}$ -gradation. Assume further that $\text{str}(g|\mathfrak{L}(F)_{(m,n)}) = \text{str}(g|\mathfrak{L}(F)_{(-m,-n)})$ for all $g \in G$, $m, n > 0$, and*

$$g \cdot f_{-1} = f_{-1}, \quad g \cdot e_{-1} = e_{-1} \quad \text{for all } g \in G.$$

Then, for all $g \in G$ and $m, n > 0$, we have

$$(7.4) \quad \text{str}(g|\mathfrak{L}(F)_{(m,n)}) = \text{str}(g|\mathfrak{L}(F)_{(-m,-n)}) \\ = \sum_{d|(m,n)} \frac{1}{d} \mu(d) \sum_{s \in T(\frac{m}{d}, \frac{n}{d})} \frac{(|s| - 1)!}{s!} \prod f_{g^d}(i+j-1)^{s_{ij}},$$

where $f_g(i) = \text{str}(g|V_i) = \text{str}(g|\mathfrak{L}_{(-1,-i)})$ for all $g \in G$ and $i = 1, 2, 3, \dots$. □

Corollary 7.2 *Suppose that the generalized supercharacters $F_g(q) = \sum_{n=-1}^{\infty} f_g(n)q^n$ are replicable with the replicates $F_g^{(a)}$ ($a \geq 1$) for all $g \in G$. Then we have*

$$(7.5) \quad \text{str}(g|\mathfrak{L}(F)_{(m,n)}) = \text{str}(g|\mathfrak{L}(F)_{(-m,-n)}) = \sum_{ad|(m,n)} \frac{1}{ad} \mu(d) f_{g^d}^{(a)} \left(\frac{mn}{a^2 d^2} \right).$$

Proof. Since the generalized supercharacter $F_g(q)$ is replicable, by (7.1), the generalized denominator identity for $g \in G$ of the Lie superalgebra $\mathfrak{L}(F)_-^{(J)}$ is equal to

$$\begin{aligned} & \prod_{m,n=1}^{\infty} \exp \left(- \sum_{k=1}^{\infty} \frac{1}{k} \text{str}(g^k|\mathfrak{L}(F)_{(-m,-n)}) p^{km} q^{kn} \right) \\ &= 1 - \sum_{i,j=1}^{\infty} f_g(i+j-1) p^i q^j \\ &= \prod_{m,n=1}^{\infty} \exp \left(- \sum_{a=1}^{\infty} \frac{1}{a} f_g^{(a)}(mn) p^{am} q^{an} \right). \end{aligned}$$

By taking the logarithm, we have

$$\begin{aligned} & \log \prod_{m,n=1}^{\infty} \exp \left(- \sum_{k=1}^{\infty} \frac{1}{k} \text{str}(g^k|\mathfrak{L}(F)_{(-m,-n)}) p^{km} q^{kn} \right) \\ &= - \sum_{m,n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k} \text{str}(g^k|\mathfrak{L}(F)_{(-m,-n)}) p^{km} q^{kn} \\ &= - \sum_{m,n=1}^{\infty} \sum_{\substack{k>0 \\ k|(m,n)}} \frac{1}{k} \text{str}(g^k|\mathfrak{L}(F)_{(-\frac{m}{k}, -\frac{n}{k})}) p^m q^n, \end{aligned}$$

and

$$\begin{aligned} & \log \prod_{m,n=1}^{\infty} \exp \left(- \sum_{a=1}^{\infty} \frac{1}{a} f_g^{(a)}(mn) p^{am} q^{an} \right) \\ &= - \sum_{m,n=1}^{\infty} \sum_{a=1}^{\infty} \frac{1}{a} f_g^{(a)}(mn) p^{am} q^{an} \\ &= - \sum_{m,n=1}^{\infty} \sum_{\substack{a>0 \\ a|(m,n)}} \frac{1}{a} f_g^{(a)} \left(\frac{mn}{a^2} \right) p^m q^n. \end{aligned}$$

It follows that

$$\sum_{\substack{k>0 \\ k|(m,n)}} \frac{1}{k} \text{str}(g^k|\mathfrak{L}(F)_{(-\frac{m}{k}, -\frac{n}{k})}) = \sum_{\substack{a>0 \\ a|(m,n)}} \frac{1}{a} f_g^{(a)} \left(\frac{mn}{a^2} \right).$$

Therefore, by Möbius inversion, we obtain

$$\text{str}(g|\mathfrak{L}(F)_{(-m,-n)}) = \sum_{ad|(m,n)} \frac{1}{ad} \mu(d) f_{g^d}^{(a)} \left(\frac{mn}{a^2 d^2} \right). \quad \square$$

8 Orbit Lie Superalgebras for the Dynkin Diagram Automorphisms of Borchers-Cartan Data

In the following two sections, we will discuss the applications of the generalized denominator identities for Dynkin diagram automorphisms of generalized Kac-Moody superalgebras. The notion of *orbit Lie superalgebras* will play an important role in our discussion. Most of our result is a generalization of the corresponding result on Kac-Moody algebras and generalized Kac-Moody algebras obtained in [13] and [14]. Let $A = (a_{ij})_{i,j \in I}$ be a Borchers-Cartan matrix of charge $\underline{m} = (m_i | i \in I)$ with a coloring matrix $C = (\theta_{ij})_{i,j \in I}$, and let $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, C)$ be the corresponding generalized Kac-Moody superalgebra.

Definition 8.1 A bijection $\sigma : I \rightarrow I$ is called a *Dynkin diagram automorphism* of the Borchers-Cartan data (A, \underline{m}, C) if it satisfies: (i) $a_{ij} = a_{\sigma(i), \sigma(j)}$ for all $i, j \in I$, (ii) $\sigma(I^{even}) \subset I^{even}$, $\sigma(I^{odd}) \subset I^{odd}$, (iii) $m_{\sigma(i)} = m_i$ for all $i \in I$.

In this paper, we will assume that the Dynkin diagram automorphism has finite order N .

Given a Dynkin diagram automorphism σ of a Borchers-Cartan data (A, \underline{m}, C) , we extend it to a Lie superalgebra automorphism of $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, C)$ by defining

$$(8.1) \quad \begin{aligned} \sigma(e_{ik}) &= e_{\sigma(i), k} \quad \text{for } i \in I, 1 \leq k \leq m_i, \\ \sigma(f_{ik}) &= f_{\sigma(i), k} \quad \text{for } i \in I, 1 \leq k \leq m_i, \\ \sigma(h_i) &= h_{\sigma(i)}, \quad \sigma(d_i) = d_{\sigma(i)} \quad \text{for } i \in I. \end{aligned}$$

It is straightforward to check that σ is indeed a Lie superalgebra automorphism of \mathfrak{g} . Let $U = U(\mathfrak{g})$ be the universal enveloping algebra of \mathfrak{g} . Then the Dynkin diagram automorphism σ of \mathfrak{g} can be extended to an algebra automorphism of $U(\mathfrak{g})$ by

$$\sigma(x_1 \cdots x_k) = \sigma(x_1) \cdots \sigma(x_k) \quad \text{for } x_i \in \mathfrak{g},$$

which we will also denote by σ .

Moreover, the Dynkin diagram automorphism σ induces an automorphism σ^* of \mathfrak{h}^* defined by

$$(8.2) \quad \sigma^*(\lambda)(h) = \lambda(\sigma^{-1}(h)) \quad \text{for } \lambda \in \mathfrak{h}^*, h \in \mathfrak{h}.$$

In particular, for all $i \in I$, we have $\sigma^*(\alpha_i) = \alpha_{\sigma(i)}$. Since σ has order N , σ^* also has order N , and we have an eigenspace decomposition

$$\mathfrak{h}^* = \bigoplus_{k=0}^{N-1} (\mathfrak{h}^*)^{(k)},$$

where $(\mathfrak{h}^*)^{(k)} = \{\lambda \in \mathfrak{h}^* \mid \sigma^*(\lambda) = e^{\frac{2\pi i k}{N}} \lambda\}$. The elements of $(\mathfrak{h}^*)^{(0)} = \{\lambda \in \mathfrak{h}^* \mid \sigma^*(\lambda) = \lambda\}$ are called *symmetric*.

Now, we will construct a generalized Kac-Moody superalgebra $\check{\mathfrak{g}} = \mathfrak{g}(\sigma)$, called the *orbit Lie superalgebra*, associated with the Dynkin diagram automorphism σ of \mathfrak{g} . Let σ be a Dynkin diagram automorphism of the Borchers-Cartan data (A, \underline{m}, C) . Since σ permutes the elements of I , I is decomposed into a disjoint union of σ -orbits: $I = \bigsqcup [i]$, where $[i] = \{\sigma^k(i) \mid k = 0, 1, \dots, N_i - 1\}$. Let $N_i = |[i]|$ denote the number of elements in the σ -orbit $[i]$. For each $i \in I$, let \bar{i} be the smallest element of the σ -orbit of i , and let $\hat{I} = \{\bar{i} \mid i \in I\}$ be the set of such representatives. Set $\hat{I}^{re} = \hat{I} \cap I^{re}$, $\hat{I}^{im} = \hat{I} \cap I^{im}$, $\hat{I}^{even} = \hat{I} \cap I^{even}$, and $\hat{I}^{odd} = \hat{I} \cap I^{odd}$.

Let $\hat{A} = \hat{A}(\sigma) = (\hat{a}_{\bar{i}, \bar{j}})_{\bar{i}, \bar{j} \in \hat{I}}$ be the square matrix indexed by \hat{I} whose entries are defined by

$$(8.3) \quad \hat{a}_{\bar{i}, \bar{j}} = \epsilon_j \sum_{k=0}^{N_j-1} a_{i, \sigma^k(j)} \quad \text{for } \bar{i}, \bar{j} \in \hat{I},$$

where $\epsilon_j = 1 - \sum_{k=1}^{N_j-1} a_{j, \sigma^k(j)}$. Note that $\epsilon_j = \epsilon_{\sigma(j)}$ for all $j \in I$.

Let

$$(8.4) \quad \check{I} = \{i \in \hat{I} \mid \epsilon_i \leq 2 \text{ for } i \in I^{re}, \epsilon_i = 1 \text{ for } i \in I^{im}\}.$$

Since $\hat{a}_{\bar{i}, \bar{i}} = \epsilon_i(a_{ii} + 1 - \epsilon_i)$, the condition on \check{I} is equivalent to $\hat{a}_{\bar{i}, \bar{i}} = a_{ii}$ for all $\bar{i} \in \hat{I}$. (The first half of the condition (8.4) was introduced in [13] and was referred to as the *linking condition*.) Also, we set $\check{A} = \check{A}(\sigma) = (\hat{a}_{\bar{i}, \bar{j}})_{\bar{i}, \bar{j} \in \check{I}}$, $\check{I}^{re} = \check{I} \cap I^{re}$, $\check{I}^{im} = \check{I} \cap I^{im}$, $\check{I}^{even} = \check{I} \cap I^{even}$, and $\check{I}^{odd} = \check{I} \cap I^{odd}$.

One can easily verify that \hat{A} is a symmetrizable Borchers-Cartan matrix. For instance, let $\hat{s}_{\bar{i}} = N_i \epsilon_i s_i$ ($\bar{i} \in \hat{I}$) and let $\hat{D} = \text{diag}(\hat{s}_{\bar{i}} \mid \bar{i} \in \hat{I})$, then $\hat{D}\hat{A}$ is symmetric.

We define the charge $\underline{\hat{m}} = (\hat{m}_{\bar{i}} \mid \bar{i} \in \hat{I})$ and the coloring matrix $\hat{C} = (\hat{\theta}_{\bar{i}, \bar{j}})_{\bar{i}, \bar{j} \in \hat{I}}$ of \hat{A} by

$$(8.5) \quad \hat{m}_{\bar{i}} = m_i, \quad \hat{\theta}_{\bar{i}, \bar{j}} = \theta_{ij} \quad \text{for } \bar{i}, \bar{j} \in \hat{I}.$$

Consider the generalized Kac-Moody superalgebra $\hat{\mathfrak{g}} = \mathfrak{g}(\hat{A}, \underline{\hat{m}}, \hat{C})$ associated with the Borchers-Cartan data $(\hat{A}, \underline{\hat{m}}, \hat{C})$. We denote by $\hat{e}_{\bar{i}, k}$, $\hat{f}_{\bar{i}, k}$, $\hat{h}_{\bar{i}}$, $\hat{d}_{\bar{i}}$ ($\bar{i} \in \hat{I}$, $k = 1, 2, \dots, m_i$) the generators of Lie superalgebra $\hat{\mathfrak{g}}$, and by $\hat{\mathfrak{h}} = \left(\bigoplus_{\bar{i} \in \hat{I}} \mathbb{C} \hat{h}_{\bar{i}} \right) \oplus \left(\bigoplus_{\bar{i} \in \hat{I}} \mathbb{C} \hat{d}_{\bar{i}} \right)$ the Cartan subalgebra of $\hat{\mathfrak{g}}$. We also denote by $\hat{\Pi} = \{\hat{\alpha}_{\bar{i}} \mid \bar{i} \in \hat{I}\}$ the set of simple roots of $\hat{\mathfrak{g}}$ and by $\hat{\Phi}$ the set of roots of $\hat{\mathfrak{g}}$. The *orbit Lie superalgebra* $\check{\mathfrak{g}} = \mathfrak{g}(\sigma)$ is defined to be the subalgebra of $\hat{\mathfrak{g}}$ generated by $\hat{e}_{\bar{i}, k}$, $\hat{f}_{\bar{i}, k}$, ($\bar{i} \in \check{I}$, $k = 1, 2, \dots, m_i$) $\hat{h}_{\bar{i}}$, $\hat{d}_{\bar{i}}$ ($\bar{i} \in \hat{I}$, $k = 1, 2, \dots, m_i$). We denote by $\check{\Pi} = \{\hat{\alpha}_{\bar{i}} \mid \bar{i} \in \check{I}\}$ the set of simple roots of $\check{\mathfrak{g}}$ and by $\check{\Phi}$ the set of roots of $\check{\mathfrak{g}}$.

As in the case of $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, C)$, there is a symmetric bilinear form on $\hat{\mathfrak{h}}^*$ satisfying $(\hat{\lambda} \mid \hat{\alpha}_{\bar{i}}) = \hat{\lambda}(\hat{s}_{\bar{i}} \hat{h}_{\bar{i}})$ for $\hat{\lambda} \in \hat{\mathfrak{h}}^*$, $\bar{i} \in \hat{I}$ (see (6.2)). Define a \mathbb{C} -linear isomorphism $\phi :$

$(\mathfrak{h}^*)^{(0)} \rightarrow \widehat{\mathfrak{h}}^*$ by

$$(8.6) \quad \begin{aligned} \phi(\lambda)(\widehat{h}_{\bar{i}}) &= \lambda(h_i), \\ \phi(\lambda)(\widehat{d}_{\bar{i}}) &= \epsilon_i^{-1} \lambda(d_i) \quad \text{for } \lambda \in (\mathfrak{h}^*)^{(0)}, \bar{i} \in \widehat{I}. \end{aligned}$$

Then we obtain the following lemma which will be used in the next section.

Lemma 8.2 *For $\lambda, \mu \in (\mathfrak{h}^*)^{(0)}$, we have*

$$(\lambda|\mu) = (\phi(\lambda)|\phi(\mu)).$$

Equivalently, for $\widehat{\lambda}, \widehat{\mu} \in \widehat{\mathfrak{h}}^$, we have*

$$(\widehat{\lambda}|\widehat{\mu}) = (\phi^{-1}(\widehat{\lambda})|\phi^{-1}(\widehat{\mu})).$$

Proof. First, observe that, $\phi^{-1}(\widehat{\alpha}_{\bar{i}}) = \epsilon_i \sum_{k=0}^{N_i-1} \alpha_{\sigma^k(i)}$ for $\bar{i} \in \widehat{I}$. Hence for $\widehat{\lambda} \in \widehat{\mathfrak{h}}^*$, we obtain

$$(\widehat{\lambda}|\widehat{\alpha}_{\bar{i}}) = \widehat{\lambda}(\widehat{s}_{\bar{i}}\widehat{h}_{\bar{i}}) = \widehat{s}_{\bar{i}}\widehat{\lambda}(\widehat{h}_{\bar{i}}),$$

and

$$\begin{aligned} (\phi^{-1}(\widehat{\lambda})|\phi^{-1}(\widehat{\alpha}_{\bar{i}})) &= (\phi^{-1}(\widehat{\lambda})|\epsilon_i \sum_{k=0}^{N_i-1} \alpha_{\sigma^k(i)}) = \epsilon_i \sum_{k=0}^{N_i-1} (\phi^{-1}(\widehat{\lambda})|\alpha_{\sigma^k(i)}) \\ &= \epsilon_i \sum_{k=0}^{N_i-1} \phi^{-1}(\widehat{\lambda})(s_{\sigma^k(i)} h_{\sigma^k(i)}) = \epsilon_i N_i \phi^{-1}(\widehat{\lambda})(s_i h_i) \\ &= \epsilon_i N_i s_i \phi^{-1}(\widehat{\lambda})(h_i) = s_i \epsilon_i N_i \widehat{\lambda}(\widehat{h}_{\bar{i}}), \end{aligned}$$

which proves our claim. \square

Observe that if $\epsilon_i = 1$, the σ -orbit of i in the Dynkin diagram of \mathfrak{g} is the disjoint union of N_i copies of A_1 diagram or $B(0,1)$ diagram. If $\epsilon_i = 2$, then N_i is even and the σ -orbit of i in the Dynkin diagram of \mathfrak{g} is the disjoint union of $N_i/2$ copies of A_2 diagram.

Next, we will show that the Weyl group \check{W} of $\check{\mathfrak{g}}$ is isomorphic to a subgroup of W . We may assume that \check{W} acts on $\widehat{\mathfrak{h}}^*$ since $\check{\mathfrak{g}}$ is a subalgebra of $\widehat{\mathfrak{g}}$. For each $\bar{i} \in \check{I}^{re}$, we denote by $\check{w}_{\bar{i}}$ the simple reflection in \check{W} and define

$$(8.7) \quad \bar{w}_{\bar{i}} = \begin{cases} \prod_{k=0}^{N_i-1} w_{\sigma^k(i)} & \text{if } \epsilon_i = 1, \\ \prod_{l=0}^{N_i/2-1} w_{\sigma^l(i)} w_{\sigma^{l+N_i/2}(i)} w_{\sigma^l(i)} & \text{if } \epsilon_i = 2 \end{cases}$$

where w_i is simple reflection in W ($i \in I^{re}$). Since $a_{i,\sigma^k(i)} = 0$ for all $k \not\equiv 0 \pmod{N_i}$ if $\epsilon_i = 1$ and $a_{i,\sigma^{N_i/2}(i)} = -1$ if $\epsilon_i = 2$, $\bar{w}_{\bar{i}}$ is well-defined. Let \bar{W} be the subgroup of W generated by $\bar{w}_{\bar{i}}$'s ($\bar{i} \in \check{I}^{re}$).

Proposition 8.3 For all $\bar{i} \in \check{I}^{re}$, we have:

- (a) $\bar{w}_{\bar{i}}^2 = 1$.
- (b) $\sigma^* \bar{w}_{\bar{i}} = \bar{w}_{\bar{i}} \sigma^*$. In particular, $(\mathfrak{h}^*)^{(0)}$ is invariant under $\bar{w}_{\bar{i}}$.
- (c) For each $\lambda \in (\mathfrak{h}^*)^{(0)}$, we have

$$\bar{w}_{\bar{i}}(\lambda) = \lambda - \epsilon_i \lambda(h_i) \sum_{k=0}^{N_i-1} \alpha_{\sigma^k(i)}.$$

- (d) For each simple reflection $\check{w}_{\bar{i}} \in \check{W}$, we have $\bar{w}_{\bar{i}} = \phi^{-1} \check{w}_{\bar{i}} \phi$.

Proof. The proofs of (a), (b) and (c) are straightforward.

For (d), let $\lambda \in (\mathfrak{h}^*)^{(0)}$. Then we have

$$\begin{aligned} \phi^{-1} \check{w}_{\bar{i}} \phi(\lambda) &= \phi^{-1}(\phi(\lambda) - \phi(\lambda)(\widehat{h}_{\bar{i}}) \widehat{\alpha}_{\bar{i}}) \\ &= \phi^{-1}(\phi(\lambda) - \lambda(h_i) \widehat{\alpha}_{\bar{i}}) \\ &= \lambda - \lambda(h_i) \phi^{-1}(\widehat{\alpha}_{\bar{i}}) \\ &= \lambda - \lambda(h_i) \epsilon_i \sum_{k=0}^{N_i-1} \alpha_{\sigma^k(i)} \\ &= \bar{w}_{\bar{i}}(\lambda), \end{aligned}$$

which proves our assertion. □

Proposition 8.4 The Weyl group \check{W} of the orbit Lie superalgebra $\check{\mathfrak{g}}$ is isomorphic to the subgroup \bar{W} of W under the map $\check{w}_{\bar{i}} \mapsto \bar{w}_{\bar{i}}$ ($\bar{i} \in \check{I}^{re}$).

Proof. See [13] or [14]. □

Remark. The original proof of 8.4 in [13] consists of a somewhat lengthy calculations. In [14], the proof was simplified and it was also proved that $\bar{W} \cong \{w \in W \mid w\sigma^* = \sigma^*w\}$.

Under the above identification of \bar{W} with \check{W} , we define the *sign function* $\bar{\varepsilon} : \bar{W} \rightarrow \{\pm 1\}$ of \bar{W} by

$$\bar{\varepsilon}(\bar{w}) = \varepsilon(\phi \bar{w} \phi^{-1}) = \varepsilon(\psi^{-1}(\bar{w})) \text{ for } \bar{w} \in \bar{W},$$

where ε denotes the sign function of the Weyl group \check{W} of the orbit Lie superalgebra $\check{\mathfrak{g}} = \mathfrak{g}(\sigma)$. Note that, even though \bar{W} is a subgroup of W , we use the sign function of \check{W} , not the sign function of W .

9 Generalized Characters of Highest Weight Modules for Dynkin Diagram Automorphisms

Let $\Lambda \in P^+$ be a dominant integral weight and let $M(\Lambda)$ be the Verma module with highest weight Λ . By construction of $M(\Lambda)$, we have

$$M(\Lambda) \cong U(\mathfrak{g})/R(\Lambda),$$

where $R(\Lambda)$ is the left ideal of $U(\mathfrak{g})$ generated by the elements e_{ik} ($i \in I, k = 1, \dots, m_i$) and $h - \Lambda(h)1$ ($h \in \mathfrak{h}$). By (8.1) and (8.2), we see that $\sigma(R(\Lambda)) = R(\sigma^*(\Lambda))$. Hence the Dynkin diagram automorphism σ induces a \mathbb{C} -linear isomorphism

$$(9.1) \quad \sigma : M(\Lambda) \rightarrow M(\sigma^*(\Lambda))$$

between Verma modules. Note that we have $\sigma(M(\Lambda)_\lambda) = M(\sigma^*(\Lambda))_{\sigma^*(\lambda)}$ for all $\lambda \leq \Lambda$.

Similarly, let $V(\Lambda)$ be the irreducible highest weight module with highest weight Λ . Then by Corollary 6.4, the Dynkin diagram automorphism induces a \mathbb{C} -linear isomorphism

$$(9.2) \quad \sigma : V(\Lambda) \rightarrow V(\sigma^*(\Lambda))$$

between irreducible highest weight modules, and we have $\sigma(V(\Lambda)_\lambda) = V(\sigma^*(\Lambda))_{\sigma^*(\lambda)}$ for all $\lambda \leq \Lambda$.

In particular, if $\Lambda \in (\mathfrak{h}^*)^{(0)}$, i.e., if $\sigma^*(\Lambda) = \Lambda$, then the Dynkin diagram automorphism σ induces \mathbb{C} -linear automorphisms on $M(\Lambda)$ and $V(\Lambda)$, and we can consider the generalized characters for σ :

$$(9.3) \quad \begin{aligned} \text{ch}_\sigma M(\Lambda) &= \sum_{\lambda \leq \Lambda} \text{tr}(\sigma|_{M(\Lambda)_\lambda}) e^\lambda, \\ \text{ch}_\sigma V(\Lambda) &= \sum_{\lambda \leq \Lambda} \text{tr}(\sigma|_{V(\Lambda)_\lambda}) e^\lambda. \end{aligned}$$

Note that we have only to take the sum over the weights λ such that $\sigma^*(\lambda) = \lambda$.

Recall that the automorphism σ^* permutes the roots of \mathfrak{g} in such a way that we have

$$\sigma^*(\Phi^\pm) = \Phi^\pm, \quad \sigma^*(\Phi_0^\pm) = \Phi_0^\pm, \quad \sigma^*(\Phi_1^\pm) = \Phi_1^\pm.$$

Thus Φ^\pm is a disjoint union of σ^* -orbits of the elements in Φ^\pm . Let S^\pm denote the set of representatives of σ^* -orbits in Φ^\pm and set

$$S_0^\pm = S^\pm \cap \Phi_0, \quad S_1^\pm = S^\pm \cap \Phi_1.$$

For each root $\alpha \in \Phi$, let N_α denote the number of elements in σ^* -orbit of α , and define

$$(9.4) \quad (\alpha) = \sum_{k=0}^{N_\alpha-1} \sigma^{*k}(\alpha) \quad \text{and} \quad \mathfrak{g}_{(\alpha)} = \bigoplus_{k=0}^{N_\alpha-1} \mathfrak{g}_{\sigma^{*k}(\alpha)}.$$

Then, by the Poincaré-Birkhoff-Witt Theorem, we have

$$U(\mathfrak{g}^\pm) \cong S(\mathfrak{g}_0^\pm) \otimes \Lambda(\mathfrak{g}_1^\pm),$$

where

$$S(\mathfrak{g}_0^\pm) \cong \bigotimes_{\alpha \in S_0^\pm} S(\mathfrak{g}_{(\alpha)}), \quad \text{and} \quad \Lambda(\mathfrak{g}_1^\pm) \cong \bigotimes_{\beta \in S_1^\pm} \Lambda(\mathfrak{g}_{(\beta)})$$

as \mathbb{C} -vector spaces. Hence the generalized (super)character of the Verma module $M(0)$ for the Dynkin diagram automorphism σ is given by

$$\begin{aligned} \text{ch}_\sigma M(0) &= \text{sch}_\sigma M(0) = \sum_{\lambda \leq 0} \text{tr}(\sigma | M(0)_\lambda) e^\lambda \\ &= \prod_{\alpha \in S_0^+} \left(\sum_{m \geq 0} \text{tr}(\sigma | S^m(\mathfrak{g}_{(-\alpha)})) e^{-m(\alpha)/N_\alpha} \right) \\ &\quad \times \prod_{\beta \in S_1^+} \left(\sum_{n \geq 0} \text{tr}(\sigma | \Lambda^n(\mathfrak{g}_{(-\beta)})) e^{-n(\beta)/N_\beta} \right) \\ &= \prod_{\alpha \in S_0^+} \exp \left(\sum_{m \geq 1} \frac{1}{m} \text{tr}(\sigma^m | \mathfrak{g}_{(-\alpha)}) e^{-m(\alpha)/N_\alpha} \right) \\ &\quad \times \prod_{\beta \in S_1^+} \exp \left(- \sum_{n \geq 1} \frac{(-1)^n}{n} \text{tr}(\sigma^n | \mathfrak{g}_{(-\beta)}) e^{-n(\beta)/N_\beta} \right) \\ &= \prod_{\alpha \in S^+} \exp \left(\sum_{m \geq 1} \frac{1}{m} \text{str}(\sigma^m | \mathfrak{g}_{(-\alpha)}) E^{-m(\alpha)/N_\alpha} \right) \\ &= \prod_{\alpha \in \Phi^+} \exp \left(\sum_{m \geq 1} \frac{1}{m N_\alpha} \text{str}(\sigma^m | \mathfrak{g}_{(-\alpha)}) E^{-m(\alpha)/N_\alpha} \right). \end{aligned}$$

Note that if ρ is a Weyl vector of the generalized Kac-Moody superalgebra \mathfrak{g} , then $\phi(\rho)$ is a Weyl vector of the orbit Lie superalgebra $\check{\mathfrak{g}}$.

Proposition 9.1 *For all $\bar{w} \in \bar{W}$, we have*

$$(9.5) \quad \bar{w} \left(e^{-\rho} \text{ch}_\sigma M(0) \right) = \bar{\varepsilon}(\bar{w}) e^{-\rho} \text{ch}_\sigma M(0).$$

Proof. It suffices to prove our assertion for the simple reflections $\bar{w}_{\check{i}} \in \bar{W}$ ($\check{i} \in \check{I}^{re}$). For each $i \in I^{re}$, let $r_i = \exp(e'_i) \exp(-f'_i) \exp(e'_i)$, where the elements e'_i , f'_i and h'_i are defined by

$$\begin{cases} e'_i = e_i, & f'_i = f_i, & h'_i = h_i & \text{if } i \in I^{even}, \\ e'_i = \frac{1}{4}[e_i, e_i], & f'_i = \frac{1}{4}[f_i, f_i], & h'_i = \frac{1}{2}h_i & \text{if } i \in I^{odd}. \end{cases}$$

Then for each $i \in I$, the elements e'_i , f'_i , and h'_i generate the subalgebra of \mathfrak{g} isomorphic to $sl(2, \mathbb{C})$, and by the same argument in [30], we can verify that $r_i \in \text{Aut}(\mathfrak{g})$ and $r_i(\mathfrak{g}_\alpha) = \mathfrak{g}_{w_i(\alpha)}$.

We first consider the case when $\epsilon_i = 1$. (Recall that, in this case, $a_{i, \sigma(i)} = 0$.) Suppose $\alpha \neq \alpha_i$ and $\alpha \neq 2\alpha_i$. If we put $\bar{r}_i = \prod_{k=0}^{N_i-1} r_{\sigma^k(i)}$, then since $a_{i, \sigma^k(i)} = 0$, we get $r_{\sigma^{k'}(i)} r_{\sigma^k(i)} = r_{\sigma^k(i)} r_{\sigma^{k'}(i)}$. It follows that

$$\text{str}(\sigma^m | \mathfrak{g}_{(-\alpha)}) = \text{str}(\bar{r}_i \sigma^m \bar{r}_i^{-1} | \mathfrak{g}_{(-\bar{w}_i \alpha)}) = \text{str}(\sigma^m | \mathfrak{g}_{(-\bar{w}_i \alpha)}).$$

If $\alpha = \alpha_i$ and $i \in I^{\text{even}}$, then, since

$$\text{str}(\sigma^m | \mathfrak{g}_{(-\alpha_i)}) = \begin{cases} N_i & \text{if } m \equiv 0 \pmod{N_i}, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\exp \left(\sum_{m=1}^{\infty} \frac{1}{m} \text{str}(\sigma^m | \mathfrak{g}_{(-\alpha_i)}) E^{-m(\alpha_i)/N_i} \right) = (1 - E^{-(\alpha_i)})^{-1},$$

and

$$\bar{w}_i(e^{-\rho}(1 - E^{-(\alpha_i)})^{-1}) = e^{-\rho+(\alpha_i)}(1 - E^{(\alpha_i)})^{-1} = -e^{-\rho}(1 - E^{-(\alpha_i)})^{-1}.$$

Similarly, if $\alpha = \alpha_i$ or $\alpha = 2\alpha_i$ for $i \in I^{\text{odd}}$, then

$$\begin{aligned} & \exp \left(\sum_{m=1}^{\infty} \frac{1}{m} \text{str}(\sigma^m | \mathfrak{g}_{(-\alpha_i)}) E^{-m(\alpha_i)/N_i} \right) \\ & \times \exp \left(\sum_{m=1}^{\infty} \frac{1}{m} \text{str}(\sigma^m | \mathfrak{g}_{(-2\alpha_i)}) E^{-m(2\alpha_i)/N_i} \right) \\ & = (1 - E^{-(\alpha_i)})(1 - E^{-(2\alpha_i)})^{-1}, \end{aligned}$$

and

$$\begin{aligned} & \bar{w}_i(e^{-\rho}(1 - E^{-(\alpha_i)})(1 - E^{-(2\alpha_i)})^{-1}) \\ & = e^{-\rho+(\alpha_i)}(1 - E^{(\alpha_i)})(1 - E^{(2\alpha_i)})^{-1} \\ & = -e^{-\rho+(\alpha_i)}(1 + e^{(\alpha_i)})(1 - e^{(2\alpha_i)})^{-1} \\ & = -e^{-\rho}(1 - E^{-(\alpha_i)})(1 - E^{-(2\alpha_i)})^{-1}. \end{aligned}$$

Next, we consider the case when $\epsilon_i = 2$. In this case, we have $a_{i, \sigma^{N_i/2}(i)} = -1$, which implies that $i \in I^{\text{even}}$. Suppose $\alpha \neq \alpha_k$ and $\alpha \neq \alpha_k + \alpha_{\sigma^{N_i/2}(k)}$ for any $k = \sigma^l(i)$ ($1 \leq l \leq N_i/2$). Set $\bar{r}_i = \prod_{l=0}^{N_i/2-1} r_{\sigma^l(i)} r_{\sigma^{l+N_i/2}(i)} r_{\sigma^l(i)}$. Since $a_{k, \sigma^{N_i/2}(k)} = -1$ implies $r_k r_{\sigma^{N_i/2}(k)} r_k = r_{\sigma^{N_i/2}(k)} r_k r_{\sigma^{N_i/2}(k)}$ (see, for example, [33]), we have $\sigma \bar{r}_i = \bar{r}_i \sigma$ and

$$\text{str}(\sigma^m | \mathfrak{g}_{(-\alpha)}) = \text{str}(\bar{r}_i \sigma^m \bar{r}_i^{-1} | \mathfrak{g}_{(-\bar{w}_i \alpha)}) = \text{str}(\sigma^m | \mathfrak{g}_{(-\bar{w}_i \alpha)}).$$

If $\alpha = \alpha_k$ or $\alpha = \alpha_k + \alpha_{\sigma^{N_i/2}(k)}$ for some $k = \sigma^l(i)$ ($1 \leq l \leq N_i/2$) (i.e. $(\alpha) = (\alpha_i)$), then

$$\exp \left(\sum_{m=1}^{\infty} \frac{1}{m} \text{str}(\sigma^m | \mathfrak{g}_{(-\alpha_i)}) E^{-m(\alpha_i)/N_i} \right) = (1 - E^{-(\alpha_i)})^{-1}$$

$$\exp \left(\sum_{m=1}^{\infty} \frac{1}{m} \text{str}(\sigma^m | \mathfrak{g}_{(-\alpha_i - \alpha_{\sigma^{N_i/2}(i)})}) E^{-2m(\alpha_i + \alpha_{\sigma^{N_i/2}(i)})/N_i} \right) = (1 + E^{-(\alpha_i)})^{-1},$$

and

$$\begin{aligned} \bar{w}_i(e^{-\rho}(1 - E^{(-\alpha_i)})^{-1}(1 + E^{(-\alpha_i)})^{-1}) \\ = -e^{-\rho}(1 - E^{(-\alpha_i)})^{-1}(1 + E^{(-\alpha_i)})^{-1}, \end{aligned}$$

which completes the proof of our assertion. \square

Proposition 9.2 *For $\Lambda \in P^+$ and $\bar{w} \in \bar{W} \subset W$, we have*

$$(9.6) \quad \bar{w}(ch_{\sigma}V(\Lambda)) = ch_{\sigma}V(\Lambda).$$

Proof. It suffices to prove (9.6) for $\bar{w} = \bar{w}_{\bar{i}}$ ($\bar{i} \in \check{I}^{re}$). Let $\pi : \mathfrak{g} \rightarrow gl(V(\Lambda))$ denote the representation of \mathfrak{g} on $V(\Lambda)$ and define

$$\bar{r}_{\bar{i}}^{\pi} = \begin{cases} \prod_{k=0}^{N_i-1} r_{\sigma^k(i)}^{\pi} & \text{if } \epsilon_i = 1, \\ \prod_{l=0}^{N_i/2-1} r_{\sigma^l(i)} r_{\sigma^{l+N_i/2}(i)} r_{\sigma^l(i)} & \text{if } \epsilon_i = 2. \end{cases}$$

By the same argument in the preceding proposition, we have $\sigma \bar{r}_{\bar{i}}^{\pi} = \bar{r}_{\bar{i}}^{\pi} \sigma$. It follows that

$$\text{tr}(\sigma | V(\Lambda)_{\lambda}) = \text{tr}(\sigma | V(\Lambda)_{\bar{w}_{\bar{i}}\lambda}),$$

which proves our assertion. \square

Now, we are in a position to state and prove the main result of this section.

Theorem 9.3 *Let σ be a Dynkin diagram automorphism of a generalized Kac-Moody superalgebra \mathfrak{g} and let $\check{\mathfrak{g}} = \mathfrak{g}(\sigma)$ be the corresponding orbit Lie superalgebra. Let $\phi : (\mathfrak{h}^*)^{(0)} \rightarrow \hat{\mathfrak{h}}^*$ be the \mathbb{C} -linear isomorphism given by (8.6), and extend it to a \mathbb{C} -linear map $\phi : \mathbb{C}[[\mathfrak{h}^*]^{(0)}] \rightarrow \mathbb{C}[[\hat{\mathfrak{h}}^*]]$ by $\phi(e^{\lambda}) = e^{\phi(\lambda)}$ for $\lambda \in (\mathfrak{h}^*)^{(0)}$. Then, for a symmetric dominant integral weight $\Lambda \in P^+ \cap (\mathfrak{h}^*)^{(0)}$, we have*

$$(9.7) \quad \phi(ch_{\sigma}V(\Lambda)) = ch\check{V}(\phi(\Lambda)),$$

where $\check{V}(\phi(\Lambda))$ is the irreducible highest weight $\check{\mathfrak{g}}$ -module with highest weight $\phi(\Lambda)$.

Proof. Using the properties of Casimir operator given in Proposition 6.2, the same argument in [31] yields

$$(9.8) \quad \text{ch}_\sigma V(\Lambda) = \sum_{\substack{\lambda \leq \Lambda \\ |\lambda + \rho|^2 = |\Lambda + \rho|^2}} c_\lambda \text{ch}_\sigma M(\lambda).$$

Since $\text{ch}_\sigma M(\lambda) = e^\lambda \text{ch}_\sigma M(0)$, by (9.8), we have

$$(9.9) \quad e^\rho R_\sigma \text{ch}_\sigma V(\Lambda) = \sum_{\substack{\lambda \leq \Lambda \\ |\lambda + \rho|^2 = |\Lambda + \rho|^2}} c_\lambda e^{\lambda + \rho},$$

where

$$R_\sigma = (\text{ch}_\sigma M(0))^{-1} = \prod_{\alpha \in \Phi^+} \exp \left(- \sum_{m \geq 1} \frac{1}{m N_\alpha} \text{str}(\sigma^m | \mathfrak{g}_{(-\alpha)}) E^{-m(\alpha)/N_\alpha} \right).$$

We may assume that the sum in the right-hand side of (9.9) is taken over the weights λ which are invariant under σ^* .

If $\bar{w}(\lambda + \rho) = \tau + \rho$, then $c_\tau = \bar{\varepsilon}(\bar{w})c_\lambda$. Choose a weight τ such that $c_\tau \neq 0$. Then $\bar{w}(\tau + \rho) \leq \Lambda + \rho$ for all $\bar{w} \in \bar{W}$. Let λ be an element of the set $\{\bar{w}(\tau + \rho) - \rho \mid \bar{w} \in \bar{W}\}$ such that $\text{ht}(\Lambda - \lambda)$ is minimal. Note that $(\lambda + \rho | \alpha_i) \geq 0$ for all $\bar{i} \in \hat{I}^{re} \setminus \check{I}^{re}$ by definition of \check{I} . If $(\lambda + \rho | \alpha_i) < 0$ for some $\bar{i} \in \check{I}^{re}$, then $\bar{w}_{\bar{i}}(\lambda + \rho) = \lambda + \rho - \epsilon_i(\lambda + \rho)(h_i)(\alpha_i)$ and $\text{ht}(\Lambda - \bar{w}_{\bar{i}}(\lambda)) < \text{ht}(\Lambda - \lambda)$, which is a contradiction. Hence, $(\lambda + \rho | \alpha_i) \geq 0$ for all $i \in I^{re}$. Moreover, since $|\Lambda + \rho|^2 = |\lambda + \rho|^2$, we have

$$(9.10) \quad 0 = |\Lambda + \rho|^2 - |\lambda + \rho|^2 = (\Lambda | \Lambda - \lambda) + (\lambda + 2\rho | \Lambda - \lambda),$$

where $\Lambda - \lambda = \sum_{i \in I} n_i \alpha_i$ lies in Q^+ .

If $i \in I^{re}$, then

$$(\lambda + 2\rho | \alpha_i) = (\lambda + \rho | \alpha_i) + (\rho | \alpha_i) > 0.$$

If $i \in I^{im}$ and $n_i \geq 1$, then

$$\begin{aligned} (\lambda + 2\rho | \alpha_i) &= (\lambda + \alpha_i | \alpha_i) \\ &= (\Lambda - \sum_{j \in I \setminus \{i\}} n_j \alpha_j - (n_i - 1)\alpha_i | \alpha_i) \\ &= (\Lambda | \alpha_i) - \sum_{j \in I \setminus \{i\}} n_j (\alpha_j | \alpha_i) - (n_i - 1)(\alpha_i | \alpha_i) \geq 0. \end{aligned}$$

It follows that $n_i = 0$ for all $i \in I^{re}$. That is, $\lambda = \Lambda - \sum_{i \in I^{im}} n_i \alpha_i$. From (9.10), for $k \in I^{im}$ with $n_k \geq 1$, we have

$$(\lambda + 2\rho | \alpha_k) = (\lambda + \alpha_k | \alpha_k) = (\Lambda | \alpha_k) - \sum_{i \in I^{im}} n'_i (\alpha_i | \alpha_k) = 0,$$

where $n'_i = n_i$ for $i \neq k$ and $n'_k = n_k - 1$. Hence we have

$$\begin{aligned} (\Lambda|\alpha_k) &= 0, \\ (\alpha_i|\alpha_k) &= 0 \text{ if } n_i \geq 1 \text{ and } i \neq k, \\ (\alpha_k|\alpha_k) &= 0 \text{ if } n_k \geq 2, \end{aligned}$$

and k lies in \check{I}^{im} . Combining the above conditions and the fact that $n_i = n_{\sigma(i)}$ for all $i \in I$, we conclude that λ has the form

$$\lambda = \Lambda - \sum_{\bar{i} \in \check{I}^{im}} n_{\bar{i}}(\alpha_i),$$

where

$$\begin{aligned} ((\alpha_i)|(\alpha_j)) &= 0 \text{ if } \bar{i} \neq \bar{j} \text{ and } n_{\bar{i}}, n_{\bar{j}} \geq 1, \\ (\Lambda|(\alpha_i)) &= 0 \text{ if } n_{\bar{i}} \geq 1, \\ ((\alpha_i)|(\alpha_i)) &= 0 \text{ if } n_{\bar{i}} \geq 2. \end{aligned}$$

Let λ, λ' be two such elements of the above form. Suppose that there exists a $\bar{w} \in \bar{W}$ such that $\bar{w}(\lambda + \rho) = \lambda' + \rho$. Then, if $\lambda \in P^+$, we have $\bar{w}(\lambda) \leq \lambda$, since $\bar{w}(\lambda)$ is also a weight of the \mathfrak{g} -module $V(\lambda)$. Furthermore, by induction on the length of \bar{w} , we can verify $\bar{w}(\rho) \leq \rho$. Hence we obtain $\bar{w}(\lambda + \rho) \leq \lambda + \rho$ and $\bar{w}(\lambda + \rho) = \lambda + \rho - \sum_{\alpha \in \Phi^+, re} n_\alpha \alpha$ with $n_\alpha \geq 0$. On the other hand, if $\bar{w}(\lambda + \rho) = \lambda' + \rho$, we must have $n_\alpha = 0$ for all $\alpha \in \Phi^+, re$. Hence $\bar{w}(\lambda + \rho) = \lambda + \rho$, which implies $\lambda = \lambda'$.

Let $\Phi^+(\Lambda)$ be the set defined in (6.12) and let $\Phi^+(\Lambda)^{(0)} = \Phi^+(\Lambda) \cap (\mathfrak{h}^*)^{(0)}$ be the set of elements $\sum_{\bar{i} \in \check{I}^{im}} n_{\bar{i}}(\alpha_i)$ such that $((\alpha_i)|(\alpha_j)) = 0$ if $n_{\bar{i}}, n_{\bar{j}} \geq 1$ for $\bar{i} \neq \bar{j}$, $(\Lambda|(\alpha_i)) = 0$ if $n_{\bar{i}} \geq 1$, and $((\alpha_i)|(\alpha_i)) = 0$ if $n_{\bar{i}} \geq 2$. So (9.9) can be written as follows:

$$e^\rho R_\sigma \text{ch}_\sigma V(\Lambda) = \sum_{\bar{w} \in \bar{W}} \bar{\epsilon}(\bar{w}) \sum_{\beta \in \Phi^+(\Lambda)^{(0)}} c_{\Lambda-\beta} e^{\bar{w}(\Lambda-\beta+\rho)}.$$

Now, it remains to determine $c_{\Lambda-\beta}$ for $\beta \in S_\Lambda^{(0)}$. First, we observe that $\text{ch}_\sigma V(\Lambda)$ contains no term of the form $e^{\Lambda - \sum_{i \in I^{im}} n_i \alpha_i}$, where $(\Lambda|\alpha_i) = 0$ for all i such that $n_i \geq 1$. So, in order to determine $c_{\Lambda-\beta}$, we have only to compute the coefficient of $e^{\Lambda-\beta}$ in R_σ . Note that $((\alpha_i)|(\alpha_j)) = 0$ implies that $\alpha_i + \alpha_j \notin \Phi$.

For each $\bar{i} \in \check{I}^{im}$, we have

$$\exp \left(- \sum_{m \geq 1} \frac{1}{m} \text{str}(\sigma^m | \mathfrak{g}_{(-\alpha_i)}) E^{-m(\alpha_i)/N_i} \right) = (1 - E^{(-\alpha_i)})^{m_i}.$$

For $\beta = \sum_{\bar{i} \in \check{I}^{im}} n_{\bar{i}}(\alpha_i)$, using the formal power series expansions

$$(1-x)^n = \sum_{k=0}^n \binom{n}{k} (-1)^k x^k$$

and

$$(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} (-1)^k x^k,$$

we obtain

$$c_{\Lambda-\beta} = \prod_{\bar{i} \in \check{I}^{even}} \binom{m_{\bar{i}}}{n_{\bar{i}}} (-1)^{n_{\bar{i}}} \prod_{\bar{i} \in \check{I}^{odd}} \binom{m_{\bar{i}} + n_{\bar{i}} - 1}{n_{\bar{i}}} (-1)^{n_{\bar{i}}} = (-1)^{|\phi(\beta)|} \varepsilon(\phi(\beta)).$$

When $\Lambda = 0$, we get

$$R_{\sigma} = \sum_{\bar{w} \in \bar{W}} \bar{\varepsilon}(\bar{w}) \sum_{\lambda \in \Phi^+(0)^{(0)}} c_{-\lambda} e^{\bar{w}(\lambda+\rho)-\rho}.$$

Since $\phi((\alpha_i)) = \hat{\alpha}_{\bar{i}}$ for $\bar{i} \in \check{I}^{im}$, by Lemma 8.2, we have $((\alpha_i)|(\alpha_j)) = 0$ if and only if $(\hat{\alpha}_{\bar{i}}|\hat{\alpha}_{\bar{j}}) = 0$ and $(\Lambda|(\alpha_i)) = 0$ if and only if $(\phi(\Lambda)|\hat{\alpha}_{\bar{i}}) = 0$. It follows that $\phi(\Phi^+(\Lambda)^{(0)}) = \check{\Phi}^+(\phi(\Lambda)) \subset \hat{\mathfrak{h}}^*$. Therefore we obtain

$$\begin{aligned} \phi(\text{ch}_{\sigma} V(\Lambda)) &= \frac{\sum_{\bar{w} \in \bar{W}} \bar{\varepsilon}(\bar{w}) \sum_{\beta \in \Phi^+(\Lambda)^{(0)}} c_{\Lambda-\beta} e^{\phi(\bar{w}(\Lambda-\beta+\rho)-\rho)}}{\sum_{\bar{w} \in \bar{W}} \bar{\varepsilon}(\bar{w}) \sum_{\gamma \in \Phi^+(0)^{(0)}} c_{-\gamma} e^{\phi(\bar{w}(-\gamma+\rho)-\rho)}} \\ &= \frac{\sum_{\check{w} \in \check{W}} \varepsilon(\check{w}) \sum_{\hat{\beta} \in \check{\Phi}^+(\phi(\Lambda))} c_{\Lambda-\phi^{-1}(\hat{\beta})} e^{\check{w}(\phi(\Lambda)-\hat{\beta}+\rho)-\rho}}{\sum_{\check{w} \in \check{W}} \varepsilon(\check{w}) \sum_{\hat{\gamma} \in \check{\Phi}^+(0)} c_{-\phi^{-1}(\hat{\gamma})} e^{\check{w}(-\hat{\gamma}+\rho)-\rho}} \\ &= \frac{\sum_{\check{w} \in \check{W}} \varepsilon(\check{w}) \sum_{\hat{\beta} \in \check{\Phi}^+(\phi(\Lambda))} (-1)^{|\hat{\beta}|} \varepsilon(\hat{\beta}) e^{\check{w}(\phi(\Lambda)-\hat{\beta}+\rho)-\rho}}{\sum_{\check{w} \in \check{W}} \varepsilon(\check{w}) \sum_{\hat{\gamma} \in \check{\Phi}^+(0)} (-1)^{|\hat{\gamma}|} \varepsilon(\hat{\gamma}) e^{\check{w}(-\hat{\gamma}+\rho)-\rho}} \\ &= \text{ch} \check{V}(\phi(\Lambda)). \end{aligned}$$

□

Corollary 9.4

$$(9.11) \quad \begin{aligned} \phi(R_{\sigma}) &= \prod_{\alpha \in \check{\Phi}^+} (1 - E^{-\alpha})^{sdim \check{\mathfrak{g}}_{\alpha}}, \\ \phi(ch_{\sigma} M(\Lambda)) &= ch \check{M}(\phi(\Lambda)). \end{aligned}$$

Proof. In the proof of Theorem 9.3, we have shown that

$$R_{\sigma} = \sum_{\bar{w} \in \bar{W}} \bar{\varepsilon}(\bar{w}) \sum_{\gamma \in \Phi^+(0)^{(0)}} c_{-\gamma} e^{\bar{w}(-\gamma+\rho)-\rho}.$$

Hence we have

$$\begin{aligned} \phi(R_{\sigma}) &= \sum_{\bar{w} \in \bar{W}} \bar{\varepsilon}(\bar{w}) \sum_{\gamma \in \Phi^+(0)^{(0)}} c_{-\gamma} e^{\phi(\bar{w}(-\gamma+\rho)-\rho)} = \sum_{\check{w} \in \check{W}} \varepsilon(\check{w}) \sum_{\hat{\gamma} \in \check{\Phi}^+(0)} c_{-\phi^{-1}(\hat{\gamma})} e^{\check{w}(-\hat{\gamma}+\rho)-\rho} \\ &= \sum_{\check{w} \in \check{W}} \varepsilon(\check{w}) \sum_{\hat{\gamma} \in \check{\Phi}^+(0)} (-1)^{|\hat{\gamma}|} \varepsilon(\hat{\gamma}) e^{\check{w}(-\hat{\gamma}+\rho)-\rho} = \prod_{\alpha \in \check{\Phi}^+} (1 - E^{-\alpha})^{sdim \check{\mathfrak{g}}_{\alpha}}. \end{aligned}$$

Since $\text{ch}_\sigma M(\Lambda) = e^\Lambda R_\sigma^{-1}$, we obtain

$$\phi(\text{ch}_\sigma M(\Lambda)) = e^{\phi(\Lambda)} \prod_{\alpha \in \check{\Phi}^+} (1 - E^{-\alpha})^{-\text{sdim } \check{\mathfrak{g}}_\alpha} = \text{ch} \check{M}(\phi(\Lambda)).$$

□

Set $\widehat{Q} = \bigoplus_{i \in I} \mathbb{Z}(\widehat{\alpha}_i / \epsilon_i N_i)$, and define a \mathbb{Z} -linear map $\widehat{\phi} : Q \rightarrow \widehat{Q}$ by

$$\widehat{\phi}(\alpha_i) = \widehat{\alpha}_i / \epsilon_i N_i \quad \text{for } i \in I, \quad k = 0, 1, \dots, N_i - 1.$$

Then $\widehat{\phi}$ is an extension of $\phi|_{Q \cap (\mathfrak{h}^*)^{(0)}}$ to Q . From now on, we will also use ϕ for $\widehat{\phi}$.

For each $\beta \in \widehat{Q}$, define the subspace $\mathfrak{g}_{[\beta]}$ of \mathfrak{g} to be

$$\mathfrak{g}_{[\beta]} = \bigoplus_{\alpha \in \phi^{-1}(\beta)} \mathfrak{g}_\alpha.$$

Clearly, $[\mathfrak{g}_{[\alpha]}, \mathfrak{g}_{[\beta]}] \subset \mathfrak{g}_{[\alpha+\beta]}$, and hence \mathfrak{g} becomes a \widehat{Q} -graded Lie superalgebra with the coloring of \widehat{Q} induced by that of Q . Note that the Dynkin diagram automorphism σ preserves the \widehat{Q} -gradation of \mathfrak{g} . With respect to this gradation, the generalized denominator identity for the Dynkin diagram automorphism σ of the generalized Kac-Moody superalgebra \mathfrak{g} is the same as

$$(9.12) \quad \prod_{\beta \in \widehat{Q}^+} \exp \left(- \sum_{m=1}^{\infty} \frac{1}{m} \text{str}(\sigma^m | \mathfrak{g}_{[-\beta]}) E^{-m\beta} \right) = 1 - \text{sch}_\sigma H(\mathfrak{g}_-).$$

On the other hand, we have

$$(9.13) \quad \begin{aligned} \phi(R_\sigma) &= \prod_{\alpha \in \Phi^+} \exp \left(- \sum_{m=1}^{\infty} \frac{1}{m N_\alpha} \text{str}(\sigma^m | \mathfrak{g}_{(-\alpha)}) E^{-m\phi(\alpha)/N_\alpha} \right) \\ &= \prod_{\alpha \in \Phi^+} \exp \left(- \sum_{m=1}^{\infty} \frac{1}{m N_\alpha} \text{str}(\sigma^m | \mathfrak{g}_{(-\alpha)}) E^{-m\phi(\alpha)} \right) \\ &= \prod_{\beta \in \widehat{Q}^+} \exp \left(- \sum_{m=1}^{\infty} \frac{1}{m} \text{str}(\sigma^m | \mathfrak{g}_{[-\beta]}) E^{-m\beta} \right). \end{aligned}$$

Therefore, by combining (9.12) and (9.13) with Corollary 9.4, we obtain:

Theorem 9.5 *The generalized denominator identity of the \widehat{Q} -graded generalized Kac-Moody superalgebra \mathfrak{g} for the Dynkin diagram automorphism σ is the same as the denominator identity of the orbit Lie superalgebra $\check{\mathfrak{g}}$.* □

Remark. (a) For the Kac-Moody algebras and generalized Kac-Moody algebras, Theorem 9.3 was proved in [13] and [14]. They also gave a list of Dynkin diagram

automorphisms of Kac-Moody algebras \mathfrak{g} of finite growth and corresponding orbit Lie algebras $\check{\mathfrak{g}}$ ([13]).

(b) Let $\mathfrak{D}_\sigma(\mathfrak{g})$ denote the generalized denominator identity of the \widehat{Q} -graded generalized Kac-Moody superalgebra \mathfrak{g} for the Dynkin diagram automorphism σ and $\mathfrak{D}(\check{\mathfrak{g}})$ the denominator identity of the orbit Lie superalgebra $\check{\mathfrak{g}}$. Then, Theorem 9.5 implies

$$(9.14) \quad \mathfrak{D}_\sigma(\mathfrak{g}) = \mathfrak{D}(\check{\mathfrak{g}}).$$

(c) For the affine Kac-Moody superalgebras introduced in [30], we have

$$\begin{aligned} \mathfrak{D}_\sigma(A^{(2)}(0, 2n-1)) &= \mathfrak{D}(B^{(1)}(0, n-1)) \quad \text{when } n \geq 2 \text{ and } \sigma \text{ has order 2,} \\ \mathfrak{D}_\sigma(C^{(2)}(3)) &= \mathfrak{D}(A^{(4)}(0, 2)) \quad \text{when } \sigma \text{ has order 2,} \\ \mathfrak{D}_\sigma(C^{(2)}(0, 2n+1)) &= \mathfrak{D}(A^{(4)}(0, 2n)) \quad \text{when } n \geq 2 \text{ and } \sigma \text{ has order 2,} \\ \mathfrak{D}_\sigma(C^{(2)}(0, 2n+2)) &= \mathfrak{D}(B^{(1)}(0, n)) \quad \text{when } n \geq 2 \text{ and } \sigma \text{ has order 2.} \end{aligned}$$

The identity (9.14) can be rewritten as

$$\prod_{\beta \in \widehat{Q}^+} \exp \left(- \sum_{m=1}^{\infty} \frac{1}{m} \text{str}(\sigma^m | \mathfrak{g}_{[-\beta]}) E^{-m\beta} \right) = \prod_{\alpha \in \widehat{Q}^+} (1 - E^{-\alpha})^{\text{sdim } \mathfrak{g}(\sigma)_\alpha},$$

where $\mathfrak{g}(\sigma) = \check{\mathfrak{g}}$ is the orbit Lie superalgebra corresponding to the Dynkin diagram automorphism σ of \mathfrak{g} . Let us consider $\mathfrak{g}(\sigma^d)$ for a nonnegative integer d . Then we have the following objects as in the case of $\mathfrak{g}(\sigma)$:

- (i) $\widehat{I}(d), \check{I}(d), N_i(d), \epsilon_i(d), \widehat{A}(d), \check{A}(d), \widehat{Q}(d) = \bigoplus_{\bar{i} \in \widehat{I}(d)} \mathbb{Z}(\widehat{\alpha}_{\bar{i}}/\epsilon_i(d)N_i(d)),$
- (ii) a \mathbb{Z} -linear map $\phi_d : Q \rightarrow \widehat{Q}(d)$, where $\phi_d(\alpha_i) = \widehat{\alpha}_{\bar{i}}/\epsilon_i(d)N_i(d)$ for $i \in I$, $k = 1, \dots, N_i(d)$.

Applying the above equation to $\mathfrak{g}(\sigma^d)$, we get

$$(9.15) \quad \prod_{\beta \in \widehat{Q}^+(d)} \exp \left(- \sum_{m=1}^{\infty} \frac{1}{m} \text{str}(\sigma^{md} | \mathfrak{g}_{[-\beta]_d}) E^{-m\beta} \right) = \prod_{\alpha \in \widehat{Q}^+(d)} (1 - E^{-\alpha})^{\text{sdim } \mathfrak{g}(\sigma^d)_\alpha},$$

where $\mathfrak{g}_{[\beta]_d} = \bigoplus_{\alpha \in \phi_d^{-1}(\beta)} \mathfrak{g}_\alpha$. Also, if we define a map τ of $\widehat{I}(d)$ by $\tau(\bar{i}) = \overline{\sigma(i)}$, then it induces a \mathbb{Z} -linear map $\phi^d : \widehat{Q}(d) \rightarrow \widehat{Q}$ such that $\phi^d \phi_d(\beta) = \phi(\beta)$ for all $\beta \in Q$. By taking ϕ^d , (9.15) can be written as

$$(9.16) \quad \prod_{\beta \in \widehat{Q}^+} \exp \left(- \sum_{m=1}^{\infty} \frac{1}{m} \text{str}(\sigma^{md} | \mathfrak{g}_{[-\beta]}) E^{-m\beta} \right) = \prod_{\alpha \in \widehat{Q}^+} (1 - E^{-\alpha})^{\text{sdim } \mathfrak{g}(\sigma^d)_{[\alpha]^d}},$$

where $\mathfrak{g}(\sigma^d)_{[\alpha]^d} = \bigoplus_{\beta \in (\phi^d)^{-1}([\alpha])} \mathfrak{g}(\sigma^d)_\beta$. By taking the logarithm on both sides, we have

$$\sum_{\beta \in \widehat{Q}^+} \sum_{m=1}^{\infty} \frac{1}{m} \text{str}(\sigma^{md} | \mathfrak{g}_{[-\beta]}) E^{-m\beta} = \sum_{\alpha \in \widehat{Q}^+} \sum_{n=1}^{\infty} \frac{1}{n} \text{sdim } \mathfrak{g}(\sigma^d)_{[\alpha]^d} E^{-n\alpha}.$$

It follows that

$$\sum_{\substack{k>0 \\ \beta=k\tau}} \frac{1}{k} \text{str}(\sigma^{dk} | \mathfrak{g}_{[-\tau]}) = \sum_{\substack{a>0 \\ \beta=a\gamma}} \frac{1}{a} \text{sdim } \mathfrak{g}(\sigma^d)_{[\gamma]^d}.$$

If we put $T_{\sigma^d}(\beta) = \sum_{a|\beta} \frac{1}{a} \text{sdim } \mathfrak{g}(\sigma^d)_{[\beta/a]^d}$ for a nonnegative integer d and $\beta \in \widehat{Q}$, then by Möbius inversion

$$\begin{aligned} \text{str}(\sigma^k | \mathfrak{g}_{[-\alpha]}) &= \sum_{d|\alpha} \frac{1}{d} \mu(d) T_{\sigma^{dk}}(\alpha/d) \\ (9.17) \qquad &= \sum_{ad|\alpha} \frac{1}{ad} \mu(d) \text{sdim } \mathfrak{g}(\sigma^{dk})_{[\alpha/ad]^{dk}}, \end{aligned}$$

where μ denotes the classical Möbius function.

Let \mathfrak{g}^σ be the fixed point subalgebra of \mathfrak{g} for a Dynkin diagram automorphism σ . Then it is also graded by \widehat{Q} and by (9.17) we get

Proposition 9.6 *For each $\beta \in \widehat{Q}$,*

$$(9.18) \qquad \text{sdim } \mathfrak{g}_\beta^\sigma = \frac{1}{|\sigma|} \sum_{k=1}^{|\sigma|} \sum_{ad|\beta} \frac{1}{ad} \mu(d) \text{sdim } \mathfrak{g}(\sigma^{dk})_{[\beta/ad]^{dk}},$$

Proof. Note that $\mathfrak{g}_\beta^\sigma$ is a representation of a finite cyclic group $\langle \sigma \rangle$. Hence $\text{dim } \mathfrak{g}_\beta^\sigma$ is the multiplicity of the trivial representation of $\langle \sigma \rangle$ in $\mathfrak{g}_\beta^\sigma$ and it is the same as the average of traces of $\tau \in \langle \sigma \rangle$ on $\mathfrak{g}_\beta^\sigma$. \square

Remark. By the similar argument in [5], we can see that \mathfrak{g}^σ is a central extension of a generalized Kac-Moody superalgebra. However, the corresponding Borcherds-Cartan matrix is not known in general. Also, the decomposition of \mathfrak{g}^σ with respect to the \widehat{Q} -gradation is coarser than its root space decomposition.

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